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## REPRESENTATIONS OF THE NORMALIZERS OF MAXIMAL TORI OF SIMPLE LIE GROUPS

By

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**Abstract.** We study the branching rule for the restriction from a complex simple Lie group  $G$  to the normalizer of a maximal torus of  $G$ . We show that the problem is reduced to the determination of the Weyl group module structures induced on the zero weight spaces of representations of semisimple Lie groups. The concrete formulas are obtained for  $SL(n, \mathbb{C})$  in terms of generalized  $q$ -binomial coefficients and Schur functions.

### 1. Introduction

The zero weight space of a representation  $V$  of a complex semisimple Lie group  $G$  naturally affords a representation of the Weyl group  $W$  of  $G$ , because  $W$  is by definition the quotient group  $N/T$ , where  $T$  is a maximal torus of  $G$  and  $N = N_G(T)$  is the normalizer of  $T$  in  $G$ . We call this representation of  $W$  the zero weight representation for  $V$ , which gives interesting relationship between the representations of  $G$  and those of  $W$ . In particular, the  $W$ -module structure of the zero weight space has deep-rooted connection to various problems in the representation theory of Lie groups such as the analysis of the plethysm and generalized exponents, etc. ([2], [9], [13]). A natural extension of this problem is to study the restriction of the whole representation of  $G$  to  $N$ . This is what we pursue in this paper. We have a similar problem if we replace  $G$  by its compact real form, which is really an equivalent problem (see §9).

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This paper is organized as follows. In §§2–5 we discuss the parametrization of irreducible representations of  $N$ . In §§6–10 we study the branching rules from  $G$  to  $N$ .

In §2, we recall Clifford’s theory on the representations of an extension group  $E$  of a group  $H$  by a finite group  $F$  over the field  $\mathbf{C}$  of complex numbers ([3]), which is a purely algebraic version of the method of little groups or Mackey machine ([19], [11]). For an irreducible representation  $\chi$  of  $H$ , define  $E_\chi$  to be the subgroup leaving the equivalence class of  $\chi$  invariant and take an irreducible representation  $\tau$  of  $E_\chi$  such that  $\tau|_H$  is a multiple of  $\chi$ . Then the representation of  $E$  induced from  $\tau$  is irreducible. Every irreducible representation of  $E$  is given in this way. Furthermore it is shown in ([3]) that  $\tau$  is the tensor product of two irreducible projective representations  $\tau_1$  and  $\tau_2$  of  $E_\chi$ , where  $\tau_1$  has the same degree as  $\chi$  and satisfies  $\tau_1(gh) = \tau_1(g)\chi(h)$  ( $g \in G, h \in H$ ), and  $\tau_2$  is the pullback of an irreducible projective representation of  $E_\chi/H$ .

If  $\chi$  can be extended to an ordinary representation (by which we mean a linear representation) of  $E_\chi$ , then these two projective representations can be replaced by ordinary representations. If  $E$  is a semidirect product of  $H$  and  $F$ , then this condition holds for all  $\chi$  and therefore all irreducible representations of  $E$  can be obtained from irreducible representations of  $H$  and subgroups of  $F$ . In §3 we study the case where  $H$  is an abelian group and give a sufficient condition for all irreducible representations  $\chi$  to be extended to ordinary representations of  $E_\chi$ . Under this condition, the equivalence classes of irreducible representations of  $E$  are parametrized by the *conjugacy classes*, under the action of  $E$ , of the pairs  $(\chi, \phi)$ , where  $\chi$  is an irreducible character of  $H$  and  $\phi$  is an irreducible character of the factor group  $E_\chi/H$ . The characters of  $E$  are given in §4.

These results hold for abstract groups and finite-dimensional representations over algebraically closed fields. Also they are valid for complex simple Lie groups and their holomorphic finite-dimensional representations, or compact simple Lie groups and their continuous finite-dimensional representations in complex vector spaces. In §5 we study the case where  $H$  is a maximal torus  $T$  of a connected complex semisimple Lie group  $G$  and  $E$  is its normalizer  $N$  in  $G$ , which is an extension of  $T$  by the Weyl group  $W$ . Although  $N$  is not a semidirect product of  $T$  and  $W$  in general, the equivalence classes of irreducible holomorphic representations of  $N$  can be parametrized by the conjugacy classes of  $(\chi, \phi)$ , where  $\chi$  is a holomorphic character of  $T$  and  $\phi$  is an irreducible representation of the parabolic subgroup  $N_\chi/T$  of  $W$ .

We show this in two ways. One method is to choose good representatives in  $N$  of the elements of  $W$  and apply the results of §2. The other is to find

a semidirect product group containing  $N$  and to apply the results of §3. The former is canonical, but sometimes the latter is convenient for concrete calculation.

In §6 we discuss the structure of the  $N$ -module  $V \downarrow_N^G$  obtained from a  $G$ -module  $V$  by restriction from  $G$  to  $N$ , by applying the results of the preceding sections for a complex simple Lie group  $G$  and the normalizer  $N$  of a maximal torus  $T$  of  $G$ . We show that the problem can be reduced to the determination of the structure of “zero weight representations” for  $V \downarrow_{L'_p}^G$ , where  $L'_p$  varies over the derived groups of the Levi parts of parabolic subgroups of  $G$ . The same results can also be formulated starting with the compact real forms of  $G$  (see §9).

In the last four sections we study in detail the case where  $G = SL(n, \mathbf{C})$  by using Young diagrams and Schur functions. We calculate the multiplicity of an irreducible representation of  $N$  in the restriction of an irreducible representation of  $G$  to  $N$  in two ways.

In §7 and §8 we apply the results of §6 and determine the zero weight representations for  $V \downarrow_{L'_p}^G$ . The multiplicities can be written in terms of Littlewood-Richardson’s coefficient, characters of parabolic subgroups of  $W$  and generalized  $q$ -binomial coefficients.

In §9 we adopt the compact group formulation and consider the unitary group  $U(n)$  and the normalizer  $N'$  of a maximal torus of  $U(n)$ . Since an element of  $N'$  is a product of a permutation matrix and a diagonal matrix, the restriction of an irreducible character of  $U(n)$  to a connected component of  $N'$  can be regarded as a function on  $T$ . This enables us to calculate the multiplicities combinatorially in terms of the Schur functions and Weyl groups. The multiplicity formula for  $Sp(n)$  on this line is obtained by the second author ([18]).

In the last section we give a series of examples of irreducible modules  $V$  for  $GL(n, \mathbf{C})$  and their weights  $\mu$  such that  $\bigoplus_{v \in W_\mu} V_v$  is irreducible as an  $N$ -module, where  $V_v$  denotes the  $v$ -weight space of  $V$  and  $W_\mu = N_\mu/T$ . For the case where  $\mu = 0$ , every irreducible representation of the symmetric group  $\mathfrak{S}_n$  can be obtained as the zero weight representation of a suitable irreducible representation of  $SL(n, \mathbf{C})$ . ([8], [9], see also [2]).

Naruse ([14]) and Nishiyama ([15]) have obtained the results in the case where  $G = GL(n, \mathbf{C})$  and  $N$  is replaced by the symmetric group of degree  $n$ , which are related to the plethysms and the representations of party algebras.

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## 2. Representations of Finite Extensions of Groups

Throughout this paper, all representations are finite dimensional defined over the field  $\mathbf{C}$  of complex numbers.

We begin with recalling Clifford's theory on the relationship of representations of a group and those of its normal subgroup ([3], see also [7] §11). We include proofs for completeness as well as to confirm some delicate points (see Remark 2.5 in particular).

The results in this section holds for abstract groups and finite-dimensional representations over algebraically closed fields. Also they are valid for complex (resp. compact) simple Lie groups and their holomorphic (resp. continuous) finite-dimensional representations in complex vector spaces which we will study in the subsequent sections (see Remark 2.7). In each of these cases, all homomorphisms and mappings are morphisms in the relevant category.

Let  $E$  be an extension of a group  $H$  by a finite group  $F$ , i.e. there is an exact sequence of groups:

$$1 \rightarrow H \rightarrow E \rightarrow F \rightarrow 1. \quad (2.1)$$

A section  $\sigma : F \rightarrow E$  is a morphism in the relevant category not necessarily preserving group multiplications. Since  $F$  is finite, there always exists a section.

The group  $E$  acts on the representations of  $H$  by

$$g \cdot \chi(h) = \chi(g^{-1}hg),$$

where  $\chi$  is a representation of  $H$ ,  $g \in E$ ,  $h \in H$ . The representation  $g \cdot \chi$  is called a conjugate representation of  $\chi$  and denoted by  $\chi^g$ . Let  $E_\chi$  be the subgroup of  $E$  leaving the equivalence class of  $\chi$  invariant:

$$E_\chi = \{g \in E \mid \chi^g \text{ is equivalent to } \chi\}.$$

We denote by  $\text{Irr}(G)$  a complete set of representatives of the equivalence classes of irreducible representations of a group  $G$  and by  $\text{Irr}(G, \alpha)$  a complete set of representatives of the equivalence classes of irreducible projective representations of  $G$  with factor set  $\alpha : G \times G \rightarrow \mathbf{C}^\times$ .

For  $\chi \in \text{Irr}(H)$  let

$$\text{Irr}(E_\chi)_\chi = \{\tau \in \text{Irr}(E_\chi) \mid \tau|_H \text{ is a multiple of } \chi\} \quad (2.2)$$

**THEOREM 2.1 (Clifford).** *Let  $\chi$  be an irreducible representation of  $H$  and  $\tau$  an irreducible representation of  $E_\chi$  such that the restriction  $\tau|_H$  of  $\tau$  to  $H$  is a multiple of  $\chi$ . Let  $\theta_{\chi,\tau}$  be the induced representation of  $E$  afforded by*

$$V_{\chi,\tau} = \bigoplus_{g \in E/E_\chi} g U_{\chi,\tau}, \quad (2.3)$$

where  $U_{\chi,\tau}$  affords  $\tau$  and the summation is taken over a complete set of coset representatives of  $E/E_\chi$ . Then  $\theta_{\chi,\tau}$  is irreducible and every irreducible representation of  $E$  is given in this way. Two irreducible representations  $\theta_{\chi,\tau}$  and  $\theta_{\chi',\tau'}$  are equivalent if and only if the pair  $(\chi',\tau')$  is equivalent to a conjugate of  $(\chi,\tau)$ : there exists an element  $g'$  of  $E$  such that  $\chi'$  is equivalent to  $\chi^{g'}$  and  $\tau'$  is equivalent to the representation  $\tau \circ \phi_{g'}^{-1}$ , where  $\phi_{g'}$  is the isomorphism of  $E_\chi$  to  $E_{\chi'}$  defined by  $g \mapsto g'gg'^{-1}$ .

An irreducible representation  $\tau$  of  $E_\chi$  decomposes as the tensor product of two irreducible projective representations of  $E_\chi$ : one has the same degree as  $\chi$ , the other is the pullback  $\hat{\sigma}$  of an irreducible projective representation  $\sigma$  of  $E_\chi/H$ . The two projective representations are given as follows. Since  $\chi^g$  is equivalent to  $\chi$  for  $g \in E_\chi$ , there exists an invertible matrix  $\tau_1(g)$  of the same degree as that of  $\chi$  such that

$$\chi^g(h) = \tau_1(g)^{-1} \chi(h) \tau_1(g) \quad (2.4)$$

for all  $h \in H$ . Since

$$\tau(h)\tau(g) = \tau(g)\tau(g^{-1}hg)$$

for  $g \in E_\chi$ ,  $h \in H$ , if we take the matrix representation such that

$$\tau(h) = \begin{pmatrix} \chi(h) & & \\ & \ddots & \\ & & \chi(h) \end{pmatrix}, \quad \text{and put } \tau(g) = \begin{pmatrix} T_{11}(g) & \cdots & T_{1k}(g) \\ \vdots & & \vdots \\ T_{k1}(g) & \cdots & T_{kk}(g) \end{pmatrix},$$

then we have

$$\chi(h)T_{ij}(g) = T_{ij}(g)\chi(g^{-1}hg),$$

and hence, together with (2.4), we have

$$\chi(h)T_{ij}(g)\tau_1(g)^{-1} = T_{ij}(g)\tau_1(g)^{-1}\chi(h).$$

By Schur's lemma, for fixed  $g$  the matrix  $T_{ij}(g)\tau_1(g)^{-1}$  is a scalar matrix:

$$T_{ij}(g) = c_{ij}(g)\tau_1(g), \quad c_{ij}(g) \in \mathbb{C}.$$

Hence we have a matrix  $\tau_2(g)$  of degree  $k$  whose  $(i, j)$ -entry is  $c_{ij}(g)$  and obtain

$$\tau(g) = \tau_1(g) \otimes \tau_2(g). \quad (2.5)$$

Consider two elements  $g, g' \in E_\chi$ . Since  $\tau_1(gg')$  and  $\tau_1(g)\tau_1(g')$  both intertwine  $\chi$ , they only differ by a scalar factor by Schur's lemma:

$$\tau_1(g)\tau_1(g') = \alpha(g, g')\tau_1(gg'), \quad \alpha(g, g') \in \mathbf{C}^\times.$$

Moreover, since

$$\tau_1(gg') \otimes \tau_2(gg') = \tau(gg') = \tau(g)\tau(g') = \tau_1(g)\tau_1(g') \otimes \tau_2(g)\tau_2(g'),$$

we have

$$\tau_2(g)\tau_2(g') = \alpha(g, g')^{-1}\tau_2(gg').$$

Hence  $\tau_1$  and  $\tau_2$  are both projective representations of  $E_\chi$ , whose factor sets are inverse to each other, and we have  $\tau = \tau_1 \otimes \tau_2$ .

Since  $T_{ij}(h) = 0$  for  $i \neq j$  and  $T_{ii}(h) = \chi(h)$ , we can assume that  $\tau_1(h) = \chi(h)$  and that  $\tau_2(h)$  is the identity for  $h \in H$ . Furthermore we can take  $\tau_1$  in such a way that the factor set  $\alpha$  depends only on the coset  $E_\chi/H$ .

Let  $\{s_f \mid f \in F\}$  be a complete set of coset representatives of  $E/H$  with  $\pi(s_f) = f$  and  $s_1 = 1$ , where  $\pi: E \rightarrow F$  is the projection. Let  $F_\chi = \pi(E_\chi)$ . Let  $\tau_1(s_1)$  be the identity matrix and fix  $\tau_1(s_f)$  ( $f \in F_\chi, f \neq 1$ ) in any way. Define

$$\tau_1(s_f h) = \tau_1(s_f)\chi(h), \quad \tau_1(h) = \chi(h), \quad f \in F_\chi, h \in H.$$

Then we have

$$\tau_1(gh) = \tau_1(g)\chi(h), \quad g \in E_\chi, h \in H, \quad (2.6)$$

and

$$\begin{aligned} \tau_1(gh)\tau_1(g'h') &= \tau_1(g)\chi(h)\tau_1(g')\chi(h') \\ &= \tau_1(g)\tau_1(g')\chi(g'^{-1}hg')\chi(h') \\ &= \alpha(g, g')\tau_1(gg')\chi(g'^{-1}hg'h') \\ &= \alpha(g, g')\tau_1(ghg'h'), \end{aligned}$$

hence we have

$$\alpha(gh, g'h') = \alpha(g, g').$$

This shows that the factor set  $\alpha$  is associated to the factor set of the coset  $E_\chi/H$ . Therefore, by (2.5), we have  $\tau_2(gh) = \tau_2(g)$  ( $g \in E_\chi, h \in H$ ) provided that  $\tau_1$

satisfies (2.6). Hence  $\tau_2$  defines a projective representation of  $E_\chi/H$ , whose factor set is inverse to the factor set of  $E_\chi/H$  given by  $\alpha$  as above.

We next show  $\alpha$  is essentially determined by  $\chi$  and the extension (2.1). We explain this in terms of non-abelian cohomology ([4], [5], [6], [16]), which is defined as follows.

For groups  $G$  and  $A$ , let  $\kappa : G \rightarrow \text{Aut}(A)/\text{Int}(A)$  be a homomorphism of abstract groups (even if  $G$  and  $A$  are Lie groups), where  $\text{Aut}(A)$  (resp.  $\text{Int}(A)$ ) is the group of automorphisms (resp. inner automorphisms) of  $A$ . Let  $Z^2(G, A, \kappa)$  denote the set of all pairs  $(\gamma, \delta)$  of mappings

$$\gamma : G \times A \rightarrow A((g, a) \mapsto \gamma(g)(a)), \quad \delta : G \times G \rightarrow A,$$

such that

$$\gamma(g) \bmod \text{Int}(A) = \kappa(g), \quad g \in G, \quad (2.7)$$

$$\gamma(g')(\gamma(g)(a)) = \delta(g, g')^{-1}(\gamma(gg')(a))\delta(g, g'), \quad (2.8)$$

$$\delta(g, g'g'')\delta(g', g'') = \delta(gg', g'')\gamma(g'')(\delta(g, g')), \quad (2.9)$$

for  $a \in A$ ,  $g, g', g'' \in G$ .

An equivalence relation is defined as follows:  $(\gamma, \delta)$  is equivalent to  $(\gamma', \delta')$  if there exists a mapping  $h : G \rightarrow A$  such that

$$\gamma'(g)(a) = h(g)^{-1}(\gamma(g)(a))h(g), \quad (2.10)$$

$$\delta'(g, g') = h(gg')^{-1}\delta(g, g')(\gamma(g')(h(g)))h(g'), \quad (2.11)$$

for  $a \in A$ ,  $g, g' \in G$ . The cohomology set  $H^2(G, A, \kappa)$  is the set of equivalence classes in  $Z^2(G, A, \kappa)$ , which is the set of equivalence classes of extensions of  $A$  by  $G$ ,

$$1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1,$$

such that there exists a section  $s : G \rightarrow E$  and  $\kappa$  is the image in  $\text{Aut}(A)/\text{Int}(A)$  of the automorphism of  $A$  given by  $a \mapsto s(g)^{-1}as(g)$ .

In our case, let  $G = F_\chi$ ,  $A = GL(V_\chi)$  and  $\kappa = 1$  (sending  $F_\chi$  to the identity), where  $\chi : H \rightarrow GL(V_\chi)$  is the irreducible representation. Let  $\beta$  be the normalized factor set of the extension (2.1) associated with the representatives  $\{s_f | f \in F\}$ :

$$s_f s_{f'} = s_{ff'} \beta(f, f'), \quad f, f' \in F, \beta(f, f') \in H,$$

$$\beta(1, f') = \beta(f, 1) = 1.$$



LEMMA 2.2. *Let  $\gamma : F_\chi \rightarrow \text{Aut}(GL(V_\chi))$  be the mapping defined by  $\gamma(f)(X) = \tau_1(s_f)^{-1}X\tau_1(s_f)$ . Then*

- (i)  $(1, \alpha^{-1}I_{V_\chi}), (\gamma, \chi \circ \beta) \in Z^2(F_\chi, GL(V_\chi), 1)$ , where  $I_{V_\chi} \in GL(V_\chi)$  is the identity transformation and  $(\alpha^{-1}I_{V_\chi})(f, f') = \alpha^{-1}(f, f')I_{V_\chi}$ .
- (ii)  $(1, \alpha^{-1}I_{V_\chi})$  is equivalent to  $(\gamma, \chi \circ \beta)$ .

PROOF. (i) Since  $\alpha^{-1}$  satisfies the cocycle condition,  $(1, \alpha^{-1}I_{V_\chi})$  satisfies (2.7), (2.8) and (2.9). Since  $\gamma(f) \in \text{Int}(GL(V_\chi))$ , (2.7) is satisfied for  $\gamma$ . For  $f, f' \in F_\chi$  and  $X \in GL(V_\chi)$ , since

$$\begin{aligned} \tau_1(s_f)\tau_1(s_{f'}) &= \alpha(f, f')\tau_1(s_f s_{f'}) \\ &= \alpha(f, f')\tau_1(s_{ff'}\beta(f, f')) \\ &= \alpha(f, f')\tau_1(s_{ff'})\chi(\beta(f, f')) \quad (\text{by (2.6)}) \end{aligned} \tag{2.12}$$

we have

$$\begin{aligned} \gamma(f')(\gamma(f)(X)) &= (\tau_1(s_f)\tau_1(s_{f'}))^{-1}X(\tau_1(s_f)\tau_1(s_{f'})) \\ &= \chi(\beta(f, f'))^{-1}(\tau_1(s_{ff'})^{-1}X\tau_1(s_{ff'}))\chi(\beta(f, f')) \\ &= ((\chi \circ \beta)(f, f'))^{-1}(\gamma(ff')(X))(\chi \circ \beta)(f, f'). \end{aligned}$$

Since  $\beta$  satisfies the cocycle condition

$$\beta(f, f'f'')\beta(f', f'') = \beta(ff', f'')s_{f''}^{-1}\beta(f, f')s_{f''},$$

we have

$$\begin{aligned} (\chi \circ \beta)(f, f'f'')(\chi \circ \beta)(f', f'') &= \chi(\beta(f, f'f''))\chi(\beta(f', f'')) \\ &= \chi(\beta(f, f'f'')\beta(f', f'')) \\ &= \chi(\beta(ff', f'')s_{f''}^{-1}\beta(f, f')s_{f''}) \\ &\stackrel{(2.4)}{=} \chi(\beta(ff', f''))\tau_1(s_{f''})^{-1}\chi(\beta(f, f'))\tau_1(s_{f''}) \\ &= (\chi \circ \beta)(ff', f'')\gamma(f'')( \chi \circ \beta)(f, f'). \end{aligned}$$

(ii) Let  $h : F_\chi \rightarrow GL(V_\chi)$  be the mapping defined by  $h(f) = \tau_1(s_f)$ . Then, for  $f \in F_\chi$ ,  $X \in GL(V_\chi)$ , we have

$$\begin{aligned} \gamma(f)(X) &= \tau_1(s_f)^{-1}X\tau_1(s_f) \\ &= h(f)^{-1}Xh(f). \end{aligned}$$

It follows from (2.12), we have

$$\chi(\beta(f, f')) = \tau_1(s_{ff'})^{-1} \alpha(f, f')^{-1} \tau_1(s_f) \tau_1(s_{f'}), \quad f, f' \in F_\chi.$$

Thus (2.10) and (2.11) are satisfied.  $\square$

It follows from Lemma 2.2 that two cocycles  $(1, \alpha^{-1}I_{V_\chi}), (\gamma, \chi \circ \beta) \in Z^2(F_\chi, GL(V_\chi), 1)$  ( $\gamma: F_\chi \rightarrow \text{Aut}(GL(V_\chi))$  is defined by  $\gamma(f)(X) = \tau_1(s_f)^{-1} X \tau_1(s_f)$ ) define the same cohomology class in  $H^2(F_\chi, GL(V_\chi), 1)$ . If  $\kappa$  is given by a homomorphism  $G \rightarrow \text{Aut}(A)$ , then  $H^2(G, A, \kappa)$  is canonically identified with  $H^2(G, Z(A), \kappa)$ , where  $Z(A)$  is the center of  $A$  ([16] 1.17). If  $A$  is abelian,  $H^2(G, A, \kappa)$  coincides with ordinary cohomology  $H^2(G, A)$ , where  $G$  acts on  $A$  via  $\kappa$ . Thus we have

$$H^2(F_\chi, GL(V_\chi), 1) = H^2(F_\chi, \mathbf{C}^\times),$$

since  $Z(GL(V_\chi)) \simeq \mathbf{C}^\times$ .

The discussion above shows that the factor set  $\alpha$  is determined by the irreducible representation  $\chi$  of  $H$  and the factor set of the extension (2.1).

The results are summarized as follows:

**THEOREM 2.3** (Clifford).

(i) Let  $\chi, \tau$  be as in the Theorem 2.1. Then

$$\tau = \tau_1 \otimes \tau_2$$

where  $\tau_1$  and  $\tau_2$  are irreducible projective representations of  $E_\chi$  such that the degree of  $\tau_1$  is same as the degree of  $\chi$  and  $\tau_1(gh) = \tau_1(g)\chi(h)$ ,  $\tau_2(gh) = \tau_2(g)$  for all  $g \in E_\chi$  and  $h \in H$ , so that  $\tau_2$  can be viewed as a projective representation of the factor group  $F_\chi = E_\chi/H$ . The representations  $\tau_1$  and  $\tau_2$  can be taken to be ordinary representations of  $E_\chi$  if there exists an ordinary representation  $\tau_1$  such that  $\tau_1(h) = \chi(h)$  for all  $h \in H$ .

(ii) Conversely, for a given  $\chi \in \text{Irr}(H)$ , there exists an irreducible projective representation  $\tau_1$  of  $E_\chi$  such that  $\tau_1(gh) = \tau_1(g)\chi(h)$  ( $g \in E_\chi, h \in H$ ) and  $\tau_1|_H = \chi$ . The factor set  $\alpha$  of  $\tau_1$  is determined by the cohomology class of a 2-cocycle  $\bar{\alpha} \in Z^2(F_\chi, \mathbf{C}^\times)$ , which is given by  $\chi$  and the extension (2.1). Let  $\tau_2$  be the pull back of an irreducible projective representation of  $F_\chi = E_\chi/H$  with factor set  $\bar{\alpha}^{-1}$ . Then  $\tau = \tau_1 \otimes \tau_2$  is an irreducible ordinary representation of  $E_\chi$  such that  $\tau|_H$  is a multiple of  $\chi$ . Furthermore if  $\tau_1$  is fixed, then there is a one-to-one correspondence between  $\text{Irr}(E_\chi)_\chi$  and  $\text{Irr}(F_\chi, \bar{\alpha}^{-1})$  by the relation that  $\tau$  is equivalent to  $\tau_1 \otimes \tau_2$ .

REMARK 2.4. If  $E$  is a semidirect product of  $H$  by  $F$ , then  $F$  is regarded as a subgroup of  $E$ . We can write an element of  $E$  uniquely in the form  $fh$ ,  $f \in F$ ,  $h \in H$ . If furthermore  $H$  is an abelian group  $A$ , then any linear character  $\chi$  can be extended to a linear character  $\tau_1$  of  $E_\chi$  by putting

$$\tau_1(fa) = \chi(a), \quad f \in F_\chi, a \in A, \quad (2.13)$$

since  $(fa)(f'a') = (ff')(a^{f'}a')$  and  $\chi(a^{f'}) = \chi^{f'}(a) = \chi(a)$  if  $f' \in F_\chi$ . Hence, by Theorem 2.3 (i), the irreducible representations  $\tau$  of  $E_\chi$  restricting to multiples of  $\chi$  on  $A$  are given, up to equivalence, by the representations  $\tau_1 \otimes \hat{\sigma}$ , where  $\hat{\sigma} = \sigma \circ \pi|_{E_\chi}$ ,  $\sigma \in \text{Irr}(F_\chi)$ . We will denote  $\tau_1 \otimes \hat{\sigma}$  by  $\chi \rtimes \sigma$  in the sequel. We have

$$\text{Irr}(E) = \{\theta_{\chi, \chi \rtimes \sigma} \mid \chi \in \text{Hom}(A, \mathbf{C}^\times)/F, \sigma \in \text{Irr}(F_\chi)\},$$

where  $\chi \in \text{Hom}(A, \mathbf{C}^\times)/F$  means, by abuse of notation, that  $\chi$  varies over a complete set of representatives of the  $F$ -orbits of linear characters of  $A$ .

REMARK 2.5. The condition (2.6) is worth special attention. We cannot replace  $\tau_1$  by another projective representation  $\tau'_1$  of  $E_\chi$  equivalent to  $\tau_1$  if  $\tau'_1$  doesn't satisfy (2.6).

For example, if  $\chi$  is one-dimensional, e.g. if  $H$  is abelian, the projective representation  $\tau_1$  of  $E_\chi$  is also one-dimensional, and hence equivalent to any one-dimensional linear representation  $\tau'_1$ , e.g., the trivial representation. However, this does not mean that  $\chi$  can be extended to a linear representation of  $E_\chi$ , unless  $\tau'_1|_H = \chi$  honestly holds. Accordingly, the condition that  $H$  is abelian does not assure that  $\tau_2$  can be taken to be linear representations.

In fact, let  $E = Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$  be the quaternion group and  $H$  its center  $Z = \langle a^2 \rangle$ , which is the cyclic group  $Z_2$  of order 2. If  $\chi$  is the trivial representation  $\mathbf{1}$  of  $Z$ , then  $E_\chi = Q_8$  and we can take  $\tau_1$  to be the trivial representation of  $Q_8$ . Since the factor group  $Q_8/Z$  is isomorphic to  $Z_2 \oplus Z_2$ ,  $\text{Irr}(Q_8/Z)$  consists of 4 one-dimensional representations, which, together with  $\tau_1$ , define 4 one-dimensional representations of  $Q_8$  by Theorem 2.3 (ii). If  $\chi$  is non-trivial representation  $-\mathbf{1}$  of  $Z$ , then  $E_\chi = Q_8$ . Since  $\text{Irr}(Q_8)$  consists of 4 one-dimensional representations obtained as above and a two-dimensional faithful representation, there is no one-dimensional representation of  $E_\chi = Q_8$  such that its restriction to  $Z$  is  $-\mathbf{1}$ .

In §5, we will study the case where  $H$  is a maximal torus  $T$  of a complex simple Lie group  $G$  and  $E$  is its normalizer  $N$  in  $G$ . Any character  $\chi$  of  $T$  can be extended to  $N_\chi$  (Theorem 5.1). This follows not only from the commutativity

of  $T$  but also from special conditions for the representatives in  $N$  of the elements of the Weyl groups.

REMARK 2.6. Theorem 2.1 and 2.3 show that the irreducible representations of  $E$  can be parametrized by the equivalence classes of pairs  $(\chi, \bar{\tau}_2)$  with  $\chi \in \text{Irr}(H)$  and  $\bar{\tau}_2 \in \text{Irr}(F_\chi, \alpha^{-1})$ , where  $\alpha \in H^2(F_\chi, \mathbf{C}^\times)$  is determined from the non-abelian cohomology class  $\beta \in H^2(F_\chi, H, \kappa)$  attached to the extension (2.1) by the condition

$$(\lambda_{V_\chi})_*(\alpha)^{-1} = \chi_*(\beta).$$

Here  $\chi : H \rightarrow GL(V_\chi)$  and  $\lambda_{V_\chi} : \mathbf{C}^\times \rightarrow GL(V_\chi)$  is the isomorphism from  $\mathbf{C}^\times$  to the center of  $GL(V_\chi)$ .

REMARK 2.7. If  $E$  and  $H$  are complex (resp. compact) Lie groups, then this parametrization nicely “restricts” to that of the irreducible holomorphic (resp. continuous) representations of  $E$  by those pairs  $(\chi, \bar{\tau}_2)$  in which  $\chi$  is holomorphic (resp. continuous). This can be verified by checking that each operation used in the proof of Theorem 2.1 and 2.3 “preserves” the kind of the representations considered. We observe this in the paragraphs below for complex Lie groups and holomorphic representations. The other case is utterly parallel.

Under our assumption, the groups  $E \supset E_\chi \supset H$  form a chain of open subgroups. A representation of such a group is holomorphic if and only if its restriction to an open subgroup is holomorphic. Also, subrepresentations, direct sums and conjugates of holomorphic representations are holomorphic. (By a conjugate we mean a conjugate of a representation of  $H$  or  $E_\chi$  by an element  $g$  of  $E$ , which is a representation of  $H$  or  $E_{g\chi}$  respectively.)

Now if  $\chi$  and  $\tau$  are holomorphic in Theorem 2.1, then the induced representation  $\theta_{\chi, \tau}$  is also holomorphic since  $\theta_{\chi, \tau}|_H$  is a direct sum of conjugates of  $\chi$ . In the other direction, if  $\theta \in \text{Irr}(E)$ , then  $\chi$  and  $\tau$  such that  $\theta \sim \theta_{\chi, \tau}$  are obtained as subrepresentations of  $\theta|_H$  and  $\theta|_{E_\chi}$ . Hence if  $\theta$  is holomorphic, then  $\chi$  and  $\tau$  are holomorphic. Note also that the equivalence of  $(\chi, \tau)$  as defined in Theorem 2.1 preserved holomorphicity.

Moreover if  $\chi$  is holomorphic, then any  $\tau \in \text{Irr}(E_\chi)_\chi$  is holomorphic since its restriction to  $H$  is a multiple of  $\chi$  by definition. Hence, in this case, Theorem 2.3 readily gives a parametrization of the irreducible holomorphic representations of  $E_\chi$  restricting to multiples of  $\chi$  on  $H$  by the irreducible projective representations of  $E_\chi/H$  with cocycle  $\bar{\alpha}^{-1}$ . Note that, in this case, any projective representation  $\tau_1$  of  $E_\chi$  which extends  $\chi$  and satisfies (2.6) is a holomorphic map from  $E_\chi$  to

$GL(V_\chi)$  since it is so on each of the cosets of  $H$ , which form a (disjoint) open covering of  $E_\chi$ ; and that the pullback  $\tau_2$  of a projective representation of  $E_\chi/H$  is also a holomorphic map since it is constant on each coset of  $H$ .

### 3. Relation to Semidirect Products

We will study the representations of normalizer  $N$  of a maximal torus of  $SL(n, \mathbb{C})$  in the last four sections of this paper. The representations are closely related to those of the normalizer  $\tilde{N}$  of a maximal torus of  $GL(n, \mathbb{C})$ . The group  $\tilde{N}$  is the semidirect product of the subgroup consisting of the diagonal matrices by the symmetric group of degree  $n$  and contains  $N$ . With this example in mind, we consider, more generally, the case where  $H$  is an abelian group and study the relationship between the representations of a semidirect product group and its subgroups.

Let us keep the notation of §2.

ASSUMPTION 3.1. Let  $\tilde{E}$  be the semidirect product of an abelian group  $\tilde{A}$  by a finite group  $F$ :

$$1 \rightarrow \tilde{A} \rightarrow \tilde{E} \xrightarrow{\tilde{\pi}} F \rightarrow 1$$

and  $E$  a subgroup of  $\tilde{E}$  satisfying  $\tilde{\pi}(E) = F$ . Putting  $\pi = \tilde{\pi}|_E$  and  $A = \ker \pi$ , we have an extension

$$1 \rightarrow A \rightarrow E \xrightarrow{\pi} F \rightarrow 1,$$

which is not split in general.

For  $a \in A$ ,  $f \in F$ , the element  $s_f^{-1}as_f$  is independent of the choice of a representative  $s_f$  with  $\pi(s_f) = f$ , because  $A$  is abelian. Thus we denote it by  $a^f$ .

Since  $E$  is a subgroup of  $\tilde{E}$ , an element of  $E$  is written in two ways:  $s_fa$  in  $E$  and  $fa'$  in  $\tilde{E}$  with  $a, a' \in A$ ,  $f \in F$ . These are related as follows. For each  $f \in F$ , there exists a unique element  $\tilde{\epsilon}_f \in \tilde{A}$  which satisfies

$$s_f = f\tilde{\epsilon}_f, \tag{3.1}$$

thus we have

$$a' = \tilde{\epsilon}_f a.$$

LEMMA 3.2. *The factor set  $\beta$  is given by*

$$\beta(f, f') = \frac{\tilde{\epsilon}_f^{f'} \tilde{\epsilon}_{f'}}{\tilde{\epsilon}_{ff'}}.$$

PROOF. By (3.1), we have

$$\begin{aligned} s_f s_{f'} &= s_{ff'} \beta(f, f') \\ &= ff' \tilde{\epsilon}_{ff'} \beta(f, f'). \end{aligned}$$

We also have

$$\begin{aligned} s_f s_{f'} &= f \tilde{\epsilon}_f f' \tilde{\epsilon}_{f'} \\ &= ff' \tilde{\epsilon}_f^{f'} \tilde{\epsilon}_{f'}. \end{aligned}$$

Thus we have

$$\beta(f, f') = \frac{\tilde{\epsilon}_f^{f'} \tilde{\epsilon}_{f'}}{\tilde{\epsilon}_{ff'}}. \quad \square$$

The action of  $E$  on the character group  $X$  of  $A$  induces an action of  $F$  on  $X$ . For a character  $\chi$  of  $A$ , let  $F_\chi$  be the stabilizer subgroup of  $\chi$  in  $F$ .

By Remark 2.4, the irreducible representations of  $\tilde{E}$  are given, up to equivalence, by the representations  $\theta_{\tilde{\chi}, \tilde{\chi} \rtimes \sigma}$  where  $\tilde{\chi} \in \text{Hom}(\tilde{\chi}, \mathbf{C}^\times)/F$  and  $\sigma \in \text{Irr}(F_\chi)$ .

**THEOREM 3.3.** *Let  $\chi$  be a linear character of  $A$ , and suppose that there exists a linear character  $\tilde{\chi}$  of  $\tilde{A}$  such that  $\tilde{\chi}|_A = \chi$  and  $F_{\tilde{\chi}} = F_\chi$ .*

- (i) *Every irreducible representation  $\theta_{\tilde{\chi}, \tilde{\chi} \rtimes \sigma}$  of  $\tilde{E}$  remains irreducible upon restriction to  $E$ .*
- (ii) *For an irreducible representation  $\theta_{\chi, \tau}$  of  $E$ , the projective representations  $\tau_1$  and  $\tau_2$  of Theorem 2.3 can be taken to be ordinary representations in the following way. Let  $\tilde{\tau}_1$  be a linear character of  $\tilde{E}_\chi$  defined as in (2.13) and  $\tau_1 = \tilde{\tau}_1|_{E_\chi}$ . Then  $\tau_1$  is a linear character of  $E_\chi$  with  $\tau_1(a) = \chi(a)$  for all  $a \in A$ . Hence  $\tau_2$  is an ordinary representation, which defines the irreducible representation  $\sigma$  of  $F_\chi = F_{\tilde{\chi}}$ . Put  $\tilde{\tau}_2 = \hat{\sigma}$  and  $\tilde{\tau} = \tilde{\tau}_1 \otimes \tilde{\tau}_2$ , where  $\hat{\sigma} = \sigma \circ \tilde{\pi}|_{\tilde{E}_\chi}$ . Then*

$$\theta_{\tilde{\chi}, \tilde{\chi} \rtimes \sigma}|_E = \theta_{\chi, \tau}. \quad (3.2)$$

- (iii) *Suppose that such  $\tilde{\chi}$  exists for every linear character  $\chi$  of  $A$ . Fix the choice of such a linear character  $\tilde{\chi}$  for every  $\chi$ . Then the representations  $\theta_{\chi, \tau} = \theta_{\tilde{\chi}, \tilde{\chi} \rtimes \sigma}|_E$ , parametrized by the pairs  $(\chi, \sigma)$  in the following way, form a complete set of representatives of the equivalence classes of irreducible representations of  $E$ :  $\chi$  varies over a complete set of representatives of*

the  $F$ -orbits of linear characters of  $A$ , and  $\tilde{\tau} = \tilde{\tau}_1 \otimes \hat{\sigma}$ , where  $\tilde{\tau}_1$  is determined from  $\tilde{\chi}$  as in (2.13) and  $\sigma$  varies over  $\text{Irr}(F_\chi)$ , a complete set of representatives of the equivalence classes of irreducible representations of  $F_{\tilde{\chi}} = F_\chi$ .

PROOF. (i) Since

$$\tilde{\tau}(f\tilde{a}) = \tilde{\chi}(a\tilde{\epsilon}_f^{-1})\tilde{\tau}(s_f), \quad f\tilde{a} \in \tilde{E}_\chi,$$

a  $\tilde{\tau}(E_\chi)$ -invariant subspace is also  $\tilde{\tau}(\tilde{E}_\chi)$ -invariant. Hence  $\tau = \tilde{\tau}|_{E_\chi}$  is irreducible and  $\theta_{\tilde{\chi}, \tilde{\tau}}|_E = \theta_{\chi, \tau}$ .

(ii) We have  $\tau_1|_A = \chi$ , since  $\tilde{\tau}_1|_{\tilde{A}} = \tilde{\chi}$ ,  $A = E_\chi \cap \tilde{A}$  and  $\chi = \tilde{\chi}|_A$ . Since  $\tau_1$  is a linear character of  $E_\chi$ ,  $\tau_2$  is also an ordinary representation by (2.5). By construction, we have  $\tilde{\tau}_1|_{E_\chi} = \tau_1$  and  $\tilde{\tau}_2|_{E_\chi} = \tau_2$ , so that  $\tilde{\tau}|_{E_\chi} = \tau$ . Hence

$$\theta_{\tilde{\chi}, \tilde{\tau}}|_E = \theta_{\chi, \tau}.$$

(iii) Now suppose that the assumption at the beginning of this Theorem holds for every linear character  $\chi$  of  $A$ . Let  $R(A, E)$  be a complete set of representatives of conjugacy classes of linear characters of  $A$  under the action of  $E$ .

It follows from Theorem 2.1 that every irreducible representation of  $E$  is equivalent to some  $\theta_{\chi, \tau}$ , where  $\chi \in R(A, E)$  and  $\tau \in \text{Irr}(E_\chi)$ . By (ii), we have  $\tau = \tau_1 \otimes (\sigma \circ \pi|_{E_\chi})$ . Here  $\tau_1$  is given by  $\tilde{\chi}$  as in (ii) and  $\sigma$  is an irreducible representation of  $F_\chi$ . Let  $\sigma'$  be an element of  $\text{Irr}(F_\chi)$  which is equivalent to  $\sigma$ . Then  $\theta_{\chi, \tau}$  is equivalent to  $\theta_{\chi, \tau_1 \otimes (\sigma' \circ \pi|_{E_\chi})}$ , since  $\tau$  is equivalent to  $\tau_1 \otimes (\sigma' \circ \pi|_{E_\chi})$ . By Theorem 2.1 again,  $\theta_{\chi, \tau_1 \otimes (\sigma \circ \pi|_{E_\chi})}$  and  $\theta_{\chi', \tau'_1 \otimes (\sigma' \circ \pi|_{E_{\chi'}})}$  ( $\chi, \chi' \in R(A, E), \sigma \in \text{Irr}(F_\chi), \sigma' \in \text{Irr}(F_{\chi'})$ ) are equivalent if and only if  $\chi = \chi'$  and  $\sigma = \sigma'$ .

Conversely, let  $\chi$  be an element of  $R(A, E)$  and  $\sigma \in \text{Irr}(F_\chi)$ . Then we obtain an irreducible representation  $\theta_{\tilde{\chi}, \tilde{\tau}}$  with  $\tilde{\tau} = \tilde{\chi} \rtimes \sigma$  as in Remark 2.4. By restricting  $\theta_{\tilde{\chi}, \tilde{\tau}}$  to  $E$ , we have an irreducible representation  $\theta_{\chi, \tau}$ .  $\square$

For  $\tilde{\chi}$  we define a function  $e_{\tilde{\chi}}$  on  $F$  by

$$e_{\tilde{\chi}}(f) = \tilde{\chi}(\tilde{\epsilon}_f). \quad (3.3)$$

COROLLARY 3.4. For  $s_f a \in E_\chi$ ,

$$\tilde{\tau}(s_f a) = e_{\tilde{\chi}}(f)\chi(a)\sigma(f), \quad (3.4)$$

where  $\sigma \in \text{Irr}(F_\chi)$  is defined as in Theorem 3.3 (ii).

PROOF. For  $f\tilde{a} \in \tilde{E}_{\tilde{\chi}}$  we have

$$\begin{aligned}\tilde{\tau}(f\tilde{a}) &= \tilde{\tau}_1(f\tilde{a}) \otimes \tilde{\tau}_2(f\tilde{a}) \\ &= \tilde{\chi}(\tilde{a})\sigma(f).\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{\tau}(s_f a) &= \tilde{\tau}(fa\tilde{\epsilon}_f) \\ &= \tilde{\tau}(fa)\tilde{\tau}(\tilde{\epsilon}_f) \\ &= \tilde{\chi}(a)\sigma(f)\tilde{\chi}(\tilde{\epsilon}_f) \\ &= e_{\tilde{\chi}}(f)\chi(a)\sigma(f),\end{aligned}$$

since  $\tilde{\chi}(a) = \chi(a)$  for  $a \in A$ . □

EXAMPLE 3.5. (1) Let  $\tilde{T}$  (resp.  $T$ ) be the group of all diagonal matrices in  $\tilde{G} = GL(n, \mathbf{C})$  (resp.  $G = SL(n, \mathbf{C})$ ) and  $\tilde{N}$  (resp.  $N$ ) its normalizer in  $\tilde{G}$  (resp.  $G$ ). The extensions

$$1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1$$

and

$$1 \rightarrow \tilde{T} \rightarrow \tilde{N} \rightarrow W \rightarrow 1$$

satisfy the conditions in Assumption 3.1, where  $W$  is the Weyl group of  $\tilde{G}$  with respect to  $\tilde{T}$  (resp. of  $G$  with respect to  $T$ ) isomorphic to  $\mathfrak{S}_n$ .

Any character  $\chi$  of  $T$  is the restriction of a character  $\tilde{\chi}$  of  $\tilde{T}$ . For a sequence of integers  $\mathbf{m} = (m_1, \dots, m_n)$ , let  $\tilde{\chi}_{\mathbf{m}} : \text{diag}(\tilde{t}_1, \dots, \tilde{t}_n) \mapsto \tilde{t}_1^{m_1} \dots \tilde{t}_n^{m_n}$  be a character of  $\tilde{T}$ . Then two characters  $\tilde{\chi} = \tilde{\chi}_{\mathbf{m}}$  and  $\tilde{\chi}' = \tilde{\chi}_{\mathbf{m}'}$  of  $\tilde{T}$  restrict to the same character of  $T$  if and only if  $\mathbf{m}' = \mathbf{m} + (k, \dots, k)$  or equivalently  $\tilde{\chi}' = \tilde{\chi} \otimes (\det)^k$ . Therefore if  $\tilde{\chi}|_T = \chi$ , then  $W_{\tilde{\chi}} = W_{\chi}$  and the assumption of Theorem 3.3 is satisfied.

For  $n = 2$ ,

$$\begin{aligned}\tilde{T} &= \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \middle| t_i \in \mathbf{C}^\times \right\}, \quad \tilde{N} = \tilde{T} \cup \left\{ \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} \middle| t_i \in \mathbf{C}^\times \right\} \\ T &= \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} \middle| t_1 \in \mathbf{C}^\times \right\}, \quad N = T \cup \left\{ \begin{pmatrix} 0 & t_1 \\ -t_1^{-1} & 0 \end{pmatrix} \middle| t_1 \in \mathbf{C}^\times \right\}.\end{aligned}$$



Let

$$\begin{aligned}\tilde{t} &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \quad \tilde{t}' = \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, \quad t' = \begin{pmatrix} 0 & t_1 \\ -t_1^{-1} & 0 \end{pmatrix}, \\ t' &= tg, \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned}$$

For an integer  $m$  let  $\chi(t) = t_1^m$  and  $\tilde{\chi}(\tilde{t}) = t_1^{m_1} t_2^{m_2}$  with  $m = m_1 - m_2$ ,  $m_i \in \mathbf{Z}$ . Then  $\tilde{\chi}|_T = \chi$ .

- (i) If  $m \neq 0$ , then  $N_\chi = T$  and  $W_\chi$  is trivial. Let  $\tau = \chi$ . Then

$$V_{\chi, \tau} = U_{\chi, \tau} \oplus gU_{\chi, \tau}.$$

Take a non-zero vector  $e_1 \in U_{\chi, \tau}$ , and put  $e_2 = ge_1 \in gU_{\chi, \tau}$ . The matrix representation of  $\theta_{\chi, \tau}$  with respect to the basis  $\{e_1, e_2\}$  has the form

$$\theta_{\chi, \tau}(t) = \begin{pmatrix} t_1^m & 0 \\ 0 & t_1^{-m} \end{pmatrix}, \quad \theta_{\chi, \tau}(t') = \begin{pmatrix} 0 & (-t_1)^m \\ t_1^{-m} & 0 \end{pmatrix}.$$

- (ii) If  $m = 0$ , then  $N_\chi = N$  and  $W_\chi = W \simeq \mathfrak{S}_2$ . Let  $\tilde{\chi}(\tilde{t}) = (t_1 t_2)^k$  and  $\tilde{\tau}_1(\tilde{t}) = \tilde{\tau}_1(\tilde{t}') = (t_1 t_2)^k$ . Define  $\tau_1 = \tilde{\tau}_1|_N$ , then  $\tau_1(t) = 1$  and

$$\tau_1(t') = \tilde{\tau}_1 \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix} \right) = \tilde{\chi} \left( \begin{pmatrix} -t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix} \right) = (-1)^k.$$

- (a) For the trivial representation of  $W_\chi \simeq \mathfrak{S}_2$ , we have  $\tau_2 = 1$  and

$$\theta_{\chi, \tau} = \tau = \tau_1 \otimes \tau_2 = \tau_1 \otimes 1.$$

Hence we have

$$\theta_{\chi, \tau}(t) = 1, \quad \theta_{\chi, \tau}(t') = (-1)^k.$$

- (b) For the sign representation of  $W_\chi \simeq \mathfrak{S}_2$ , we have  $\tau_2(t) = 1$ ,  $\tau_2(t') = -1$ . In this case we have

$$\theta_{\chi, \tau}(t) = 1, \quad \theta_{\chi, \tau}(t') = \tau_1(t')\tau_2(t') = (-1)^{k+1}.$$

We next consider the representations of  $\tilde{N}$ .

- (iii) If  $m_1 \neq m_2$ , then  $\tilde{N}_\chi = \tilde{T}$ . Let  $\tilde{\tau} = \tilde{\chi}$ . Then

$$V_{\tilde{\chi}, \tilde{\tau}} = U_{\tilde{\chi}, \tilde{\tau}} \oplus gU_{\tilde{\chi}, \tilde{\tau}}.$$

Take a non-zero element  $\tilde{e}_1 \in U_{\tilde{\chi}, \tilde{\tau}}$ , and put  $\tilde{e}_2 = g\tilde{e}_1$ . Then we have the matrix representation with respect to the basis  $\{e_1, e_2\}$

$$\theta_{\tilde{\chi}, \tilde{\tau}}(\tilde{t}) = \begin{pmatrix} t_1^{m_1} t_2^{m_2} & 0 \\ 0 & t_1^{m_2} t_2^{m_1} \end{pmatrix}, \quad \theta_{\tilde{\chi}, \tilde{\tau}}(\tilde{t}') = \begin{pmatrix} 0 & (-t_1)^{m_1} t_2^{m_2} \\ t_1^{m_2} (-t_2)^{m_1} & 0 \end{pmatrix}.$$

The restriction  $\theta_{\tilde{\chi}, \tilde{\tau}}|_N$  is the representation in (i).

- (iv) If  $m_1 = m_2 = k$ , then  $\tilde{N}_{\tilde{\chi}} = \tilde{N}$  and  $W_{\tilde{\chi}} = W \simeq \mathfrak{S}_2$ . Then  $\tilde{\tau}_1(\tilde{t}) = \tilde{\chi}(\tilde{t}) = (t_1 t_2)^k$  and

$$\tilde{\tau}_1(\tilde{t}') = \tilde{\tau}_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) = \tilde{\chi}\left(\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) = (t_1 t_2)^k.$$

- (a) For the trivial representation of  $W_{\tilde{\chi}} \simeq \mathfrak{S}_2$ , we have  $\tilde{\tau}_2 = 1$  and  $\theta_{\tilde{\chi}, \tilde{\tau}} = \tilde{\tau}_1 \otimes \tilde{\tau}_2$ . Hence

$$\theta_{\tilde{\chi}, \tilde{\tau}}(\tilde{t}) = \theta_{\tilde{\chi}, \tilde{\tau}}(\tilde{t}') = (t_1 t_2)^k.$$

The restriction  $\theta_{\tilde{\chi}, \tilde{\tau}}|_N$  is the representation in (ii)(a).

- (b) For the sign representation of  $W_{\tilde{\chi}} \simeq \mathfrak{S}_2$ , we have  $\theta_{\tilde{\chi}, \tilde{\tau}}(\tilde{t}) = (t_1 t_2)^k$  and

$$\begin{aligned} \theta_{\tilde{\chi}, \tilde{\tau}}(\tilde{t}') &= \theta_{\tilde{\chi}, \tilde{\tau}}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) \\ &= \tilde{\tau}_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) \otimes \tilde{\tau}_2\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) \\ &= -(t_1 t_2)^k \end{aligned}$$

Hence  $\theta_{\tilde{\chi}, \tilde{\tau}}|_N$  is the representation in (ii)(b).

- (2) We may realize the symplectic group  $Sp(2n, \mathbf{C})$  as the set of matrices  $X$  of  $GL(2n, \mathbf{C})$  satisfying  ${}^t X J X = J$ , where  $J$  is the matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let  $T$  be the group of all diagonal matrices in  $Sp(2n, \mathbf{C})$  and  $N$  the normalizer of  $T$  in  $Sp(2n, \mathbf{C})$ . The Weyl group  $W = N/T$ , faithfully represented as a group of automorphisms of  $T$  acting by conjugation, is generated by the permutations of the indices  $\{1, 2, \dots, n\}$  of the elements  $t = \text{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$  (which permute  $t_1, \dots, t_n$  and  $t_1^{-1}, \dots, t_n^{-1}$  in a parallel manner) and the transposition that exchange  $t_n$  and  $t_n^{-1}$ . This naturally gives an embedding  $\iota: W \hookrightarrow \mathfrak{S}_{2n}$  as the permutations that commute with  $(1 \ 2n)(2 \ 2n-1) \cdots (n \ n+1)$ .

Let  $\tilde{T}$  be the group of all diagonal matrices in  $GL(2n, \mathbb{C})$ . The group  $W$  can be realized as a subgroup of  $GL(2n, \mathbb{C})$  consisting of the permutation matrices representing the elements of  $\iota(W) \subset \mathfrak{S}_{2n}$ . Let  $\tilde{N}$  be the subgroup of  $GL(2n, \mathbb{C})$  generated by  $\tilde{T}$  and  $W$ , which is isomorphic to the semidirect product of  $\tilde{T}$  by  $W$ . Then  $N$  is a subgroup of  $\tilde{N}$ , and  $(W, T, N, \tilde{T}, \tilde{N})$  satisfies Assumption 3.1. For a character  $\chi$  of  $T$  given by  $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$ , define a character  $\tilde{\chi}$  of  $\tilde{T}$  by  $\tilde{\chi}(\tilde{t}) = \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$ . Then we have  $\tilde{\chi}|_T = \chi$  and  $W_{\tilde{\chi}} = W_{\chi}$ .

(3) The orthogonal group  $SO(m, \mathbb{C})$  is realized as the set of matrices  $X$  of  $SL(m, \mathbb{C})$  satisfying  ${}^tXJX = J$ , where  $J$  is the matrix

$$J = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } m = 2n + 1, \quad J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad \text{for } m = 2n.$$

Let  $T$  be the group of all diagonal matrices in  $SO(m, \mathbb{C})$  and  $N$  the normalizer of  $T$  in  $SO(m, \mathbb{C})$ . If  $m = 2n + 1$ , the Weyl group  $W = N/T$ , as in (2), is generated by the permutations of the indices  $\{1, 2, \dots, n\}$  of the elements  $t = \text{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}, 1)$  of  $T$  (which permute  $t_1, \dots, t_n$  and  $t_1^{-1}, \dots, t_n^{-1}$  in a parallel manner) and the transposition that exchange  $t_n$  and  $t_n^{-1}$ . If  $m = 2n$ , the Weyl group is generated by the permutations of the indices  $\{1, 2, \dots, n\}$  of the elements  $t = \text{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$  of  $T$  (which permute  $t_1, \dots, t_n$  and  $t_1^{-1}, \dots, t_n^{-1}$  in a parallel manner) and products of even numbers of transpositions that exchange  $t_i$  and  $t_i^{-1}$ . This naturally gives an embedding  $\iota: W \rightarrow \mathfrak{S}_{2n}$  as the permutations that commute with  $(1 \ 2n)(2 \ 2n-1) \cdots (n \ n+1)$ .

Let  $\tilde{T}$  be the group of all diagonal matrices in  $GL(m, \mathbb{C})$ . The group  $W$  can be realized as a subgroup of  $GL(m, \mathbb{C})$  consisting of the permutation matrices representing the elements of  $\iota(W) \subset \mathfrak{S}_{2n}$ . Let  $\tilde{N}$  be the subgroup of  $GL(m, \mathbb{C})$  generated by  $\tilde{T}$  and  $W$ , which is isomorphic to the semidirect product of  $\tilde{T}$  by  $W$ . Then  $N$  is a subgroup of  $\tilde{N}$ , and  $(W, T, N, \tilde{T}, \tilde{N})$  satisfies Assumption 3.1. For a character  $\chi$  of  $T$  given by  $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$ , define a character  $\tilde{\chi}$  of  $\tilde{T}$  by  $\tilde{\chi}(\tilde{t}) = \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$ . Then we have  $\tilde{\chi}|_T = \chi$  and  $W_{\tilde{\chi}} = W_{\chi}$ .

#### 4. Formula for Irreducible Characters of Finite Extensions of Abelian Groups

In this section, we calculate the character  $\psi_{\chi, \tau}$  of  $\theta_{\chi, \tau}$  for the case where  $E$  is an extension of an abelian group  $A$  by a finite group  $F$ :

$$1 \rightarrow A \rightarrow E \xrightarrow{\pi} F \rightarrow 1.$$

The notation is as in the preceding sections. Let  $\tau$  be an irreducible representation of  $E_\chi$  whose restriction to  $A$  is a multiple of  $\chi$ . Let  $\varphi_{\chi,\tau}$  be the character of  $\tau$ . By (2.3), we have

$$\psi_{\chi,\tau}(g) = \sum_{\substack{g' \in E/E_\chi \\ g'^{-1}gg' \in E_\chi}} \varphi_{\chi,\tau}(g'^{-1}gg'),$$

where the summation is taken over a complete set of coset representatives of  $E/E_\chi$ . Since  $E/E_\chi = F/F_\chi$ ,

$$\psi_{\chi,\tau}(g) = \sum_{\substack{f \in F/F_\chi \\ f^{-1}\pi(g)f \in F_\chi}} \varphi_{\chi,\tau}(\tilde{f}^{-1}g\tilde{f}),$$

where the summation is taken over a complete set of coset representatives of  $F/F_\chi$  and  $\tilde{f}$  is an element of  $\pi^{-1}(f)$ .

For  $g \in E$ , put

$$I(g) = \{f \in F \mid f^{-1}\pi(g)f \in F_\chi\}.$$

Then  $I(g)F_\chi = I(g)$  and

$$\psi_{\chi,\tau}(g) = \frac{1}{|F_\chi|} \sum_{f \in I(g)} \varphi_{\chi,\tau}(\tilde{f}^{-1}g\tilde{f}), \quad (4.1)$$

where  $\tilde{f}$  is an element of  $\pi^{-1}(f)$ .

Let  $K_f$  be the conjugacy class of  $f \in F$  in  $F$ . If  $k_0 \in F$  and  $k_0^{-1}\pi(g)k_0 = f_0 \in K_{\pi(g)} \cap F_\chi$ , then, for  $k \in F$ ,  $k^{-1}\pi(g)k = f_0$  if and only if  $kk_0^{-1}$  is an element of the centralizer  $Z_F(\pi(g))$  of  $\pi(g)$  in  $F$ .

Let  $\Phi_f : F \rightarrow F$  be the mapping given by  $k \mapsto k^{-1}fk$ . Note that  $\text{Im } \Phi_{\pi(g)} = K_{\pi(g)}$ . We have

$$I(g) = \Phi_{\pi(g)}^{-1}(F_\chi) = \bigsqcup_{f_0 \in K_{\pi(g)} \cap F_\chi} \Phi_{\pi(g)}^{-1}(f_0) \quad (\text{disjoint union}) \quad (4.2)$$

and, for each  $f_0 \in K_{\pi(g)} \cap F_\chi$ , we have

$$\Phi_{\pi(g)}^{-1}(f_0) = Z_F(\pi(g))k_0 \quad \text{if we pick any } k_0 \in \Phi_{\pi(g)}^{-1}(f_0). \quad (4.3)$$

Now we have the following.

PROPOSITION 4.1. *Let  $g \in E$ .*

(i) *For each  $f_0 \in F_\chi \cap K_{\pi(g)}$ , fix an element  $k_0 \in F$  with  $f_0 = k_0^{-1}\pi(g)k_0$ , then*

$$\psi_{\chi, \tau}(g) = \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_{\pi(g)}} \sum_{x \in Z_F(\pi(g))k_0} \varphi_{\chi, \tau}(\tilde{x}^{-1}g\tilde{x}),$$

*where  $\varphi_{\chi, \tau}$  is the character of  $\tau$  and  $\tilde{x}$  is an element of  $\pi^{-1}(x)$ .*

(ii) *Suppose  $E$  is a semidirect product of  $A$  by  $F$  and  $\tau$  is given by  $\chi \rtimes \sigma$  (see Remark 2.4). Let  $g = fa \in E$  with  $f \in F$  and  $a \in A$ . Then we have*

$$\psi_{\chi, \tau}(fa) = \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \phi_\sigma(f_0) \left( \sum_{x \in \Phi_f^{-1}(f_0)} \chi(x^{-1}ax) \right),$$

*where  $\phi_\sigma$  is the character of  $\sigma$ .*

PROOF. (i) This follows from (4.1), (4.2) and (4.3).

(ii) By (4.1), (4.2), we have

$$\psi_{\chi, \tau}(fa) = \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \sum_{x \in \Phi_f^{-1}(f_0)} \varphi_{\chi, \tau}(x^{-1}(fa)x).$$

Since  $x^{-1}(fa)x = (x^{-1}fx)(x^{-1}ax) = f^x a^x$ , we have

$$\varphi_{\chi, \tau}(x^{-1}(fa)x) = \chi(a^x) \phi_\sigma(f^x)$$

and hence

$$\begin{aligned} \psi_{\chi, \tau}(fa) &= \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \sum_{x \in \Phi_f^{-1}(f_0)} \chi(a^x) \phi_\sigma(f^x) \\ &= \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \sum_{x \in \Phi_f^{-1}(f_0)} \chi(a^x) \phi_\sigma(f_0) \\ &= \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \phi_\sigma(f_0) \left( \sum_{x \in \Phi_f^{-1}(f_0)} \chi(a^x) \right) \end{aligned} \quad \square$$

EXAMPLE 4.2. Let  $T$  be the group of all diagonal matrices in  $GL(n, \mathbf{C})$  and  $N$  the normalizer of  $T$  in  $GL(n, \mathbf{C})$ . Then  $W = N/T$  is isomorphic to the symmetric group  $\mathfrak{S}_n$  of degree  $n$  and  $N$  is a semidirect product of  $T$  by  $\mathfrak{S}_n$ :

$$1 \rightarrow T \rightarrow N \rightarrow W = \mathfrak{S}_n \rightarrow 1$$

- (i) For  $t = \text{diag}(t_1, \dots, t_n) \in T$ , let  $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$ , where  $m_1, \dots, m_n$  are distinct integers. Then  $W_\chi$  is trivial and the trivial representation is the only irreducible representation of  $W_\chi$ . Hence  $\tau = \chi \rtimes \mathbf{1}$  (see Remark 2.4). Since  $W_\chi \cap K_w = \emptyset$  for any non-trivial element  $w$  of  $W$ . Hence  $\psi_{\chi, \tau}(wt) = 0$  for  $w \in W$ ,  $w \neq 1$ ,  $t \in T$ . For  $w = 1$ , we have  $W_\chi \cap K_1 = \{1\}$  and

$$\Phi_1^{-1}(1) = W = \mathfrak{S}_n.$$

Hence we have

$$\begin{aligned} \psi_{\chi, \tau}(wt) &= \sum_{w \in \mathfrak{S}_n} \chi(t^w) \\ &= \sum_{w \in \mathfrak{S}_n} t_{w(1)}^{m_1} \cdots t_{w(n)}^{m_n}. \end{aligned}$$

- (ii) Let  $\chi(t) = (t_1 \cdots t_n)^m$ . Then  $W_\chi = \mathfrak{S}_n$  and  $W_\chi \cap K_w = K_w$  for all  $w \in \mathfrak{S}_n$ . Hence we have

$$\psi_{\chi, \tau}(wt) = \frac{1}{n!} \sum_{u \in K_w} \phi_\sigma(u) \sum_{v \in \Phi_w^{-1}(u)} \chi(t^v).$$

## 5. Case of the Normalizers of Maximal Tori

Let  $G$  be a connected complex simple Lie group. We apply the results of preceding sections to the case where  $A$  is a maximal torus  $T$  of  $G$ ,  $E$  the normalizer  $N$  of  $T$  in  $G$  and  $F$  the Weyl group  $W$  of  $G$ :

$$1 \rightarrow T \rightarrow N \xrightarrow{\pi} W \rightarrow 1.$$

In the following sections we write  $\mu$  for a weight of a representation of  $G$  with respect to  $T$  and regard  $\mu$  as a linear character of  $T$ . Thus, for the sake of brevity, we use the letter  $\mu$ , instead of  $\chi$  which is used for an irreducible representation of  $H$  or  $A$  in the previous sections, for a linear character of  $T$  when  $H$  or  $A$  is a maximal torus  $T$  of  $G$ .

When  $G$  is the classical group  $SL(n, \mathbf{C})$ ,  $Sp(2n, \mathbf{C})$  or  $SO(m, \mathbf{C})$ , if we take  $\tilde{T}$  and  $\tilde{N}$  as in Example 3.5, then  $(W, T, N, \tilde{T}, \tilde{N})$  satisfies Assumption 3.1 and the assumption of Theorem 3.3 (iii). Thus we can apply the results of §3. In general we can extend a linear character  $\mu$  to a linear character  $\tau_1$  of  $N_\mu$  by choosing good representatives in  $N$  of the elements of  $W$  and apply the results of §2.

We first recall some results about the representatives in  $N$  of the elements of  $W$  ([17] 9.3).

Let  $\Phi$  be the root system of  $G$  with respect to  $T$  and  $(,)$  the inner product induced on the character group of  $T$  or the weight lattice by the Killing form. Let  $B$  be a Borel subgroup containing  $T$ ,  $\Pi = \{\alpha_1, \dots, \alpha_l\}$  the corresponding basis of  $\Phi$  and  $\Phi^+$  the set of positive roots. For each  $\alpha \in \Phi$ , let  $u_\alpha$  be an isomorphism of  $\mathbf{C}$  into  $G$  such that

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad t \in T, x \in \mathbf{C}. \quad (5.1)$$

We can choose  $u_\alpha$  and  $u_{-\alpha}$  for each  $\alpha \in \Phi^+$  in such a way that

$$n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$$

is an element of  $N$  and represents the reflection  $s_\alpha \in W$  associated to  $\alpha$ . Then  $n_\alpha^2$  lies in  $T$ . We denote it by  $t_\alpha$ , which is given by  $\alpha^\vee(-1)$ , where  $\alpha^\vee$  is the coroot with  $\langle \alpha, \alpha^\vee \rangle = 2$ .

The normalizer  $N$  is generated by  $T \cup \{n_{\alpha_1}, \dots, n_{\alpha_l}\}$  and determined by the relations:

$$tt' = t't, \quad t, t' \in T \quad (5.2)$$

$$n_\alpha^2 = \alpha^\vee(-1), \quad \alpha \in \Pi \quad (5.3)$$

$$n_\alpha t n_\alpha^{-1} = s_\alpha(t), \quad t \in T, \alpha \in \Pi \quad (5.4)$$

$$\underbrace{n_{\alpha_i} n_{\alpha_j} n_{\alpha_i} \cdots}_{m_{ij}} = \underbrace{n_{\alpha_j} n_{\alpha_i} n_{\alpha_j} \cdots}_{m_{ij}} \quad (5.5)$$

where  $m_{ij}$  is the order of  $s_{\alpha_i} s_{\alpha_j}$ .

For  $w \in W$ , let  $w = s_{\gamma_1} \cdots s_{\gamma_k}$  be a reduced expression of  $w$  with  $\gamma_i \in \Pi$ . Then the element  $n_{\gamma_1} \cdots n_{\gamma_k}$  of  $N$  is independent of the choice of the reduced expression of  $w$ . Denote this by  $n_w$ . Hence we have a set of representatives  $\{n_w \mid w \in W\}$  in  $N$  of the elements of  $W$  provided that we fix  $\{u_\alpha \mid \alpha \in \Phi\}$  and a basis  $\Pi$  of  $\Phi$ .

**THEOREM 5.1.** (i) *Let  $\mu$  be a linear character of  $T$  and  $N_\mu$  the stabilizer of  $\mu$  in  $N$ . Let  $\tau_1$  be the mapping of  $N_\mu$  to  $\mathbf{C}$  defined by  $\tau_1(n_w t) = \mu(t)$ ,  $w \in W_\mu$ ,  $t \in T$ . Then  $\tau_1$  is a linear character of  $N_\mu$  satisfying  $\tau_1|_T = \mu$ .*

*Let  $\Lambda^+$  denote the set of the linear characters of  $T$  which are dominant with respect to  $\Pi$ . If  $\mu \in \Lambda^+$  and  $\sigma \in \text{Irr}(W_\mu)$ , then we write  $\tau(\mu, \sigma)$  for the representation  $\tau_1 \otimes \tau_2$  of  $N_\mu$ , where  $\tau_1$  is defined from  $\mu$  as in (i) and  $\tau_2 = \hat{\sigma} = \sigma \circ (\pi|_{N_\mu})$ , i.e.*

$$\tau(\mu, \sigma)(n_w t) = \mu(t)\sigma(w).$$

(ii) The representations  $\tau(\mu, \sigma)$  of  $N_\mu$  just defined above, where  $\sigma$  varies over  $\text{Irr}(W_\mu)$ , form a complete set of representatives of the equivalence classes of irreducible representations of  $N_\mu$  whose restrictions to  $T$  are multiples of  $\mu$ . Moreover, the representations  $\theta_{\mu, \tau(\mu, \sigma)}$  of  $N$  with  $\mu \in \Lambda^+$  and  $\sigma \in \text{Irr}(W_\mu)$  form a complete set of representatives of the equivalence classes of irreducible representations of  $N$ .

If  $\tau \in \text{Irr}(N_\mu)_\mu$  (see §2), then there exists a unique element  $\sigma \in \text{Irr}(W_\mu)$  such that  $\tau = \tau(\mu, \sigma)$  in the parametrization currently discussed (using the representatives  $\{n_w \mid w \in W\}$ ).

PROOF. (i) Since  $\mu$  is dominant,  $W_\mu$  is a standard parabolic subgroup. Thus  $N_\mu$  is determined by the relations (5.2)~(5.5) for  $\Pi_\mu = \{\alpha \in \Pi \mid (\alpha, \mu) = 0\}$ . Thus we have to check

$$\begin{aligned} \tau_1(n_\alpha)^2 &= \tau_1(\alpha^\vee(-1)), \quad \alpha \in \Pi_\mu \\ \tau_1(n_\alpha)\tau_1(t)\tau_1(n_\alpha)^{-1} &= \tau_1(s_\alpha(t)), \quad t \in T, \alpha \in \Pi_\mu \\ \underbrace{\tau_1(n_{\alpha_i})\tau_1(n_{\alpha_j})\tau_1(n_{\alpha_i}) \cdots}_{m_{ij}} &= \underbrace{\tau_1(n_{\alpha_j})\tau_1(n_{\alpha_i})\tau_1(n_{\alpha_j}) \cdots}_{m_{ij}}, \quad \alpha_i, \alpha_j \in \Pi_\mu. \end{aligned}$$

The last equation is obvious since  $\tau_1(n_\alpha) = 1$  for  $\alpha \in \Pi_\mu$ . Since  $\langle \alpha^\vee, \mu \rangle = 0$ , we have

$$\tau_1(\alpha^\vee(-1)) = \mu(\alpha^\vee(-1)) = (-1)^{\langle \alpha^\vee, \mu \rangle} = 1,$$

which is equal to  $\tau_1(n_\alpha)^2$ . Since  $\tau_1(n_\alpha) = 1$ , we have

$$\tau_1(n_\alpha)\tau_1(t)\tau_1(n_\alpha)^{-1} = \tau_1(t) = \mu(t).$$

On the other hand, we have

$$\tau_1(s_\alpha(t)) = \mu(s_\alpha(t)) = \mu(t),$$

since  $s_\alpha \in W_\chi$ .

(ii) By (i) every character  $\mu$  of  $T$  can be extended to a character  $\tau_1$  of  $N_\mu$ . Since every  $W$ -orbit of linear characters of  $T$  contains a unique element of  $\Lambda^+$ , the statement follows from Theorem 2.1 and 2.3.  $\square$

The proof of Theorem 5.1 (i) is based on the special properties of the representatives  $\{n_w \mid w \in W\}$ . The next proposition gives a necessary and sufficient condition for the existence of  $\tau_1$  for a finite extension of a complex algebraic torus:



PROPOSITION 5.2. *Let  $E$  be an extension of a complex algebraic torus  $A$  by a finite group  $F$ :*

$$1 \rightarrow A \rightarrow E \xrightarrow{\pi} F \rightarrow 1.$$

*For a linear character  $\chi$  of  $A$ , there exists a linear character  $\tau_1$  of  $E_\chi$  such that  $\tau_1|_A = \chi$  if and only if there is a complete set  $\{e_f\}_{f \in F_\chi}$  of coset representatives of  $E_\chi/A$  such that  $e_f e_{f'} e_{ff'}^{-1} \in \ker \chi$  for any  $f, f' \in F_\chi$ .*

PROOF. The result is obvious if  $\chi$  is trivial. Suppose  $\chi$  is not trivial and hence  $\text{Im } \chi = \mathbf{C}^\times$ . Denote by  $\Gamma(F_\chi)_\chi$  the set of complete sets  $\{e_f\}_{f \in F_\chi}$  of coset representatives of  $E_\chi/A$  satisfying the condition  $e_f e_{f'} e_{ff'}^{-1} \in \ker \chi$ ,  $f, f' \in F_\chi$  and by  $\text{Hom}(E_\chi, \mathbf{C}^\times)_\chi$  the set of linear characters  $\tau_1$  of  $E_\chi$  with  $\tau_1|_A = \chi$ . For  $\{e_f\}_{f \in F_\chi} \in \Gamma(F_\chi)_\chi$ , let  $\tau_1$  be the mapping of  $E_\chi$  to  $\mathbf{C}^\times$  defined by  $\tau_1(e_f a) = \chi(a)$ ,  $a \in A$ . Since

$$\tau_1(e_f a) \tau_1(e_{f'} a') \tau_1(e_{ff'} a a')^{-1} = \chi(a) \chi(a') \chi(a a')^{-1} = 1,$$

we have  $\tau_1 \in \text{Hom}(E_\chi, \mathbf{C}^\times)_\chi$ . This induces a mapping  $\Phi : \Gamma(F_\chi)_\chi \rightarrow \text{Hom}(E_\chi, \mathbf{C}^\times)_\chi$ .

Conversely given  $\tau_1 \in \text{Hom}(E_\chi, \mathbf{C}^\times)_\chi$ , we have  $\pi^{-1}(f) \cap \ker \tau_1 \neq \emptyset$ , since  $\tau_1(\pi^{-1}(f)) = \mathbf{C}^\times$  for every  $f \in F_\chi$ . Taking an element  $e_f \in \pi^{-1}(f) \cap \ker \tau_1$  for every  $f \in F_\chi$ , we have  $\{e_f\}_{f \in F_\chi} \in \Gamma(F_\chi)_\chi$ , since  $e_f e_{f'} e_{ff'}^{-1} \in A$  and  $\chi(e_f e_{f'} e_{ff'}^{-1}) = \tau_1(e_f e_{f'} e_{ff'}^{-1}) = 1$ . Hence  $\Phi$  is surjective.  $\square$

REMARK 5.3. The set of representatives  $\{n_w \mid w \in W\}$  satisfies the condition in Proposition 5.2.

REMARK 5.4. (i) We may choose  $\{u_\alpha \mid \alpha \in \Phi\}$  in such a way that every element  $n_\alpha$  ( $\alpha \in \Phi$ ) lies in a compact real form of  $G$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and  $\mathfrak{h}$  the Cartan subalgebra corresponding to  $T$ . For the root space decomposition  $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ , let  $\{H_\alpha, X_\alpha \mid H_\alpha \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Phi\}$  be a Chevalley basis. Then

$$\mathfrak{g}_c = \sum_{\alpha \in \Pi} \mathbf{R}(\sqrt{-1}H_\alpha) + \sum_{\alpha \in \Phi^+} \mathbf{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Phi^+} \mathbf{R}\sqrt{-1}(X_\alpha + X_{-\alpha})$$

is a compact real form. Let  $G_c$  be the compact real form of  $G$  corresponding to  $\mathfrak{g}_c$ ,  $T_c = T \cap G_c$  and  $N_c$  its normalizer in  $G_c$ . Then  $W = N/T = N_c/T_c$ .

Let us take

$$u_\alpha(x) = \exp(xX_\alpha).$$

Then (5.1)~(5.5) are satisfied ([17] 8.1, 9.3).

Let

$$X' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a Chevalley basis of  $\mathfrak{sl}(2, \mathbb{C})$  and  $u'_\pm : \mathbb{C} \rightarrow SL(2, \mathbb{C})$  the mapping defined by

$$u'_+(x) = \exp(xX') = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad u'_-(x) = \exp(xY') = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Then we have

$$n' = u'_+(1)u'_-(-1)u'_+(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is an element of the compact real form  $SU(2)$  of  $SL(2, \mathbb{C})$ .

For  $\alpha \in \Phi^+$ , there exists an isomorphism, sending  $H', X', Y'$  to  $H_\alpha, X_\alpha, X_{-\alpha}$  respectively, of  $\mathfrak{sl}(2, \mathbb{C})$  to the subalgebra of  $\mathfrak{g}$  generated by  $H_\alpha, X_\alpha, X_{-\alpha}$ . This induces a homomorphism  $\varphi$  of  $SL(2, \mathbb{C})$  to  $G$  such that  $\varphi(SU(2)) \subset G_c$ . Since

$$\begin{aligned} \varphi(n') &= \varphi(\exp(X') \exp(-Y') \exp(X')) \\ &= \exp(X_\alpha) \exp(-X_{-\alpha}) \exp(X_\alpha) \\ &= n_\alpha, \end{aligned}$$

we have  $n_\alpha \in G_c$ . Hence the same statement of Theorem 5.1 is hold for a compact real form of  $G$ .

(ii) By (5.1) we have

$$tu_\alpha(x)t^{-1}u_\alpha(x)^{-1} = u_\alpha((\alpha(t) - 1)x).$$

Hence the image of  $u_\alpha$  is in the commutator subgroup of  $G$  ([17] 7.3).

## 6. Branching from $G$ to $N$ : Reduction to Zero Weight Representations

The notations are as in §5. In this section we consider the restriction of a holomorphic representation of  $G$  to  $N$ .

Let  $\rho$  be an irreducible representation of  $G$  afforded by  $V$ . Let  $V_\mu$  be the weight space of  $T$  on  $V$  corresponding to a dominant weight  $\mu$ . Let  $N_\mu$  and  $W_\mu$  be the stabilizer subgroups of  $\mu$  in  $N$  and  $W$  respectively.

Put  $\Phi_\mu = \{\alpha \in \Phi \mid (\alpha, \mu) = 0\}$ . Let  $P$  be the parabolic subgroup of  $G$  corresponding to  $\Pi_\mu = \Pi \cap \Phi_\mu$  and  $G_P$  its Levi subgroup, which is reductive. The commutator subgroup  $G'_P$  of  $G_P$  is a semisimple Lie subgroup whose root system can be identified with  $\Phi_\mu$ .

Let

$$\rho|_{G_P} = \sum_{v \in \text{Irr}(G_P)} c_v^\rho v$$

be the decomposition of the restriction of  $\rho$  to  $G_P$  into irreducible representations of  $G_P$ .

LEMMA 6.1. *If the weight  $\mu$  is a weight of an irreducible representation  $v$  in  $\rho|_{G_P}$ , then  $\mu$  is the only weight of  $v$  that  $W_\mu$  fixes.*

PROOF. Suppose  $W_\mu$  fixes a weight  $\mu'$  of  $v$ . Let  $v$  be afforded by  $V^v$ . By the irreducibility of  $v$ ,  $V^v$  is spanned by a weight vector of weight  $\mu$  under the action of the Lie algebra  $\mathfrak{g}'_P$  of  $G'_P$ . Then  $\mu'$  must be in the form

$$\mu + \sum_{\alpha \in \Pi_\mu} m_\alpha \alpha.$$

Thus  $W_\mu$  fixes  $\sum_{\alpha \in \Pi_\mu} m_\alpha \alpha (\in \mathbf{Z}\Phi_\mu)$ , which must be 0. Thus we have  $\mu = \mu'$ .  $\square$

Put

$$\rho_\mu = \sum_{v \in M_\mu} c_v^\rho v, \tag{6.1}$$

where  $M_\mu$  is the set of irreducible representations of  $G_P$  appearing in  $\rho|_{G_P}$  of which  $\mu$  is a weight.

PROPOSITION 6.2. *The weight space  $V_\mu$  of weight  $\mu$  in  $\rho$  is the zero weight space of  $\rho_\mu$  as a  $G'_P$ -module.*

PROOF. Since the zero weights of  $G'_P$ -module are those that are fixed by the action of the Weyl group  $W_\mu$  of  $G'_P$ ,  $V_\mu$  is the zero weight space of  $\rho_\mu$  by Lemma 6.1.  $\square$

Since the Weyl group acts on the zero weight space,  $V_\mu$  can be regarded as a  $W_\mu$ -module. Let

$$v_0 = \sum_{\sigma \in \text{Irr}(W_\mu)} d_{v,\sigma}^0 \sigma \quad (6.2)$$

be the decomposition of the zero weight space of  $v|_{G'_p}$  into the irreducible representations of  $W_\mu$ . Then  $(\rho_\mu)_0$  is decomposed as a  $W_\mu$ -module:

$$\begin{aligned} (\rho_\mu)_0 &= \sum_{v \in M_\mu} c_v^\rho v_0 \\ &= \sum_{\substack{v \in M_\mu \\ \sigma \in \text{Irr}(W_\mu)}} c_v^\rho d_{v,\sigma}^0 \sigma. \end{aligned} \quad (6.3)$$

If we take the complete set of representatives

$$\{\theta_{\mu,\tau(\mu,\sigma)} \mid \mu \in \Lambda^+, \sigma \in \text{Irr}(W_\mu)\}$$

of the equivalence classes of irreducible representations of  $N$ , as given in Theorem 5.1 using  $\{n_w \mid w \in W\}$ , then we have the decomposition of the restriction of  $\rho$  to  $N$ .

**THEOREM 6.3.** *Let  $G$  be a connected complex simple Lie group and  $T, N, W$  as in §5. If  $\rho$  is an irreducible representation of  $G$ , then the restriction of  $\rho$  to  $N$  decomposes as*

$$\rho|_N = \sum_{\mu \in \Lambda_p^+} \sum_{\sigma \in \text{Irr}(W_\mu)} d_{\mu,\tau(\mu,\sigma)} \theta_{\mu,\tau(\mu,\sigma)},$$

where  $\Lambda_p^+$  is the set of dominant weights of  $\rho$  and

$$d_{\mu,\tau(\mu,\sigma)} = \sum_{v \in M_\mu} c_v^\rho d_{v,\sigma}^0$$

with  $M_\mu, c_v^\rho$  defined in (6.1) and  $d_{v,\sigma}^0$  in (6.2).

**PROOF.** Let  $U$  be a  $(\rho_\mu)_0(W_\mu)$ -invariant subspace of  $V_\mu$  affording the irreducible representation  $\sigma \in \text{Irr}(W_\mu)$ . Note that  $N_\mu$  is generated by  $T$  and the elements  $n_w$  with  $w \in W_\mu$  and since  $n_w \in G'_p$  for all  $w \in W_\mu$  by Remark 5.4 (ii), the action of  $\rho(n_w)$  on  $V_\mu$  is the same as that of  $(\rho_\mu)_0(w)$ . It follows that  $U$  is also

invariant under  $\rho(N_\mu)$  and affords the representation of  $\tau(\mu, \sigma) \in \text{Irr}(N_\mu)_\mu$  by Theorem 5.1 (i). Therefore we have

$$V_\mu \simeq \bigoplus_{\sigma \in \text{Irr}(W_\mu)} U_{\mu, \tau(\mu, \sigma)}^{\oplus d_{\mu, \tau(\mu, \sigma)}}$$

as an  $N_\mu$ -module with the numbers  $d_{\mu, \tau(\mu, \sigma)}$  defined as in the statement.

Then by comparing the construction of the induced representation (2.3) and

$$\rho(N)V_\mu = \bigoplus_{g \in N/N_\mu} \rho(g)V_\mu,$$

we have

$$\rho(N)V_\mu \simeq \bigoplus_{\sigma \in \text{Irr}(W_\mu)} V_{\mu, \tau(\mu, \sigma)}^{\oplus d_{\mu, \tau(\mu, \sigma)}}.$$

The results follows. □

There are two extremal cases.

EXAMPLE 6.4. (i) Let  $\mu$  be the highest weight of  $\rho$  and  $\rho' = d\rho$  the differential of  $\rho$ . Let  $\alpha \in \Phi^+$  be a root such that  $(\mu, \alpha) = 0$  and  $\mathfrak{a}$  the subalgebra of  $\mathfrak{g}$  generated by  $X_\alpha, H_\alpha, X_{-\alpha}$ . (see Remark 5.4(i)). By the representation theory of  $sl(2, \mathbb{C})$ , we have  $\dim \rho'(\mathfrak{a})v_\mu = (\mu, \alpha) + 1 = 1$  for a highest weight vector  $v_\mu$  and hence  $\rho'(X_\alpha)v_\mu = \rho'(X_{-\alpha})v_\mu = 0$ . Since  $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ , we have  $\rho(n_\alpha)v_\mu = v_\mu$ . It follows from Theorem 5.1 that  $\rho(N)V_\mu$  affords  $\theta_{\mu, \tau(\mu, 1)}$ .

(ii) If  $N_\mu = T$ , e.g.  $\mu$  is regular, then  $W_\mu$  is trivial, and hence we have

$$\rho(N_\mu)V_\mu = V_\mu = \mathbf{C}_\mu^{\oplus \dim V_\mu},$$

where  $\mathbf{C}_\mu$  is an irreducible  $T$ -module with weight  $\mu$ . Thus  $\rho(N)V_\mu$  affords

$$\theta_{\mu, \tau(\mu, 1)}^{\oplus \dim V_\mu}.$$

We next consider the case where  $N$  is a subgroup of some semidirect product group  $\tilde{N}$  of a complex torus  $\tilde{T}$  by  $W$  such that  $T \subset \tilde{T}$  and the assumption of Theorem 3.3(iii) is satisfied: for any irreducible character  $\mu$  of  $T$ , there exists an irreducible character  $\tilde{\mu}$  of  $\tilde{T}$  satisfying  $\tilde{\mu}|_T = \mu$  and  $W_{\tilde{\mu}} = W_\mu$ .

Having chosen such  $\tilde{\mu}$  for each  $\mu$ , it follows from Theorem 3.3(iii) that a complete set of representatives of the equivalence classes of irreducible representations of  $N$  is given by  $\{\theta_{\tilde{\mu}, \tilde{\mu} \rtimes \sigma}|_N \mid \mu \in \Lambda^+, \sigma \in \text{Irr}(W_\mu)\}$ .

Let  $\mu \in \Lambda^+$  and  $\sigma' \in \text{Irr}(W_\mu)$ . Then it follows from Theorem 3.3 that there is a unique  $\sigma \in \text{Irr}(W_{\tilde{\mu}})$  (recall our assumption that  $W_{\tilde{\mu}} = W_\mu$ ) such that the representation  $\tilde{\mu} \rtimes \sigma$  of  $\tilde{N}_{\tilde{\mu}}$  restricts to the representation  $\tau(\mu, \sigma')$  of  $N_\mu$ , so that  $\theta_{\tilde{\mu}, \tilde{\mu} \rtimes \sigma}|_N = \theta_{\mu, \tau(\mu, \sigma')}$ . We want to compare this  $\sigma$  with  $\sigma'$ .

Recall that  $\tilde{\mu} \rtimes \sigma = \tilde{\tau}_1 \otimes \hat{\sigma}$  where  $\tilde{\tau}_1(w\tilde{t}) = \tilde{\mu}(\tilde{t})$  ( $w \in W_{\tilde{\mu}} = W_\mu, \tilde{t} \in \tilde{T}$ ) and  $\hat{\sigma} = \sigma \circ (\tilde{\pi}|_{\tilde{N}_{\tilde{\mu}}})$  (Remark 2.4), and that  $\tau(\mu, \sigma') = \tau_1 \otimes \hat{\sigma}'$  where  $\tau_1(n_w t) = \mu(t)$  ( $w \in W_\mu, t \in T$ ) and  $\hat{\sigma}' = \sigma' \circ (\pi|_{N_\mu})$ . Since  $\tilde{\mu} \rtimes \sigma|_{N_\mu} = \tau(\mu, \sigma')$ , namely  $(\tilde{\tau}_1|_{N_\mu}) \otimes \hat{\sigma} = \tau_1 \otimes \hat{\sigma}'$ , the difference between  $\sigma$  and  $\sigma'$  is determined by the difference between  $\tilde{\tau}_1|_{N_\mu}$  and  $\tau_1$ . We have

$$\tilde{\tau}_1(n_w t) \tau_1(n_w t)^{-1} = \tilde{\tau}_1(w \tilde{\epsilon}_w t) \tau_1(n_w t)^{-1} = \tilde{\mu}(\tilde{\epsilon}_w t) \mu(t)^{-1} = \tilde{\mu}(\tilde{\epsilon}_w),$$

which we have defined to be  $e_{\tilde{\mu}}(w)$  ((3.3)). Note that the function  $e_{\tilde{\mu}}|_{W_\mu}$  is a linear character since, by putting  $t = 1$  we have

$$e_{\tilde{\mu}}(w) = \tilde{\tau}_1(n_w) \tau_1(n_w)^{-1} = \tilde{\tau}_1(n_w),$$

and if  $w, w' \in W_\mu$  then

$$n_w n_{w'} n_{w'}^{-1} \in \ker \tau_1 \cap T = \ker \mu \subset \ker \tilde{\mu} \subset \ker \tilde{\tau}_1.$$

Hence we have

$$\tilde{\tau}_1(n_w t) = e_{\tilde{\mu}}(w) \tau_1(n_w t), \quad w \in W_\mu, t \in T,$$

namely

$$\tilde{\tau}_1|_{N_\mu} = \widehat{e_{\tilde{\mu}}|_{W_\mu}} \otimes \tau_1.$$

By what we remarked just above, this shows that  $\sigma' = \sigma \otimes (e_{\tilde{\mu}}|_{W_\mu})$ .

Summarizing the above, we have the following:

**THEOREM 6.5.** *Let  $G$  be a connected complex simple Lie group and  $T, N, W$  as in §5. Let  $\tilde{N}$  be a semidirect product of a complex torus  $\tilde{T}$  by  $W$  such that  $\tilde{T}$  contains  $T$  and Assumption 3.1 is satisfied. Suppose that the assumption of Theorem 3.3(iii) is satisfied for  $\tilde{N}, \tilde{T}, N, T, W$  and fix the choice of an irreducible character  $\tilde{\mu}$  of  $\tilde{T}$  for every irreducible character  $\mu$  of  $T$  with  $\tilde{\mu}|_T = \mu$  and  $W_{\tilde{\mu}} = W_\mu$ . If  $\theta_{\tilde{\mu}, \tilde{\mu} \rtimes \sigma}|_N = \theta_{\mu, \tau(\mu, \sigma')}$ , then*

$$\sigma' = \sigma \otimes (e_{\tilde{\mu}}|_{W_\mu}).$$

If  $\rho$  is an irreducible representation of  $G$ , then the restriction of  $\rho$  to  $N$  decomposes as

$$\rho|_N = \sum_{\mu \in \Lambda_p^+} \sum_{\sigma \in \text{Irr}(W_\mu)} d_{\mu, \tau(\mu, \sigma)} \theta_{\mu, \tau(\mu, \sigma)},$$

where  $\Lambda_p^+$  is the set of dominant weights of  $\rho$  and

$$d_{\mu, \tilde{\mu} \times \sigma|_{N_\mu}} = d_{\mu, \tau(\mu, \sigma \otimes (e_{\tilde{\mu}}|_{W_\mu}))} = \sum_{v \in M_\mu} c_v^\rho d_{v, \sigma \otimes (e_{\tilde{\mu}}|_{W_\mu})}^0$$

with  $M_\mu$ ,  $c_v^\rho$  defined in (6.1) and  $d_{v, \sigma}^0$  in (6.2).

EXAMPLE 6.6. Let  $G = SL(n, \mathbb{C})$ . We consider the situation as in Example 3.5. For  $w \in \mathfrak{S}_n$ , let  $n_w \in N$  be a representative as in §5 whose non-zero entries are  $\pm 1$ :

$$n_w = \tilde{n}_w \tilde{t}_w,$$

where

$$\tilde{t}_w = \text{diag}(\epsilon_1, \dots, \epsilon_n), \quad \epsilon_i = (-1)^{\#\{j | 1 \leq j \leq n, i < j, w(i) > w(j)\}}$$

and  $\tilde{n}_w$  is the permutation matrix such that

$$\tilde{n}_w^{-1} \text{diag}(t_1, \dots, t_n) \tilde{n}_w = \text{diag}(t_{w(1)}, \dots, t_{w(n)}).$$

Then we have (see (3.1))

$$\tilde{\epsilon}_w = \tilde{t}_w.$$

Furthermore since

$$\begin{aligned} \det(n_w) &= \det(\tilde{n}_w) \det(\tilde{t}_w) \\ &= \text{sgn}(w) \det(\tilde{t}_w) \end{aligned}$$

and  $\det(n_w) = 1$ , we have

$$\det(\tilde{t}_w) = \text{sgn}(w).$$

For a character  $\mu$  of  $T$ ,

$$\mu(t) = \prod_{i=1}^n t_i^{m_i},$$

we have chosen  $\tilde{\mu}$  by setting

$$\tilde{\mu}(\tilde{t}) = \left( \prod_{i=1}^n \tilde{t}_i^{m_i} \right) \det(\tilde{t})^k$$

for some integer  $k$ . In particular if  $\mu$  is trivial, then  $G'_p = G$ ,  $N_\mu = N$ ,  $W_\mu = \mathfrak{S}_n$  and  $\tilde{\mu} = \det(\tilde{t})^k$ . Then we have

$$e_{\tilde{\mu}}(w) = \tilde{\mu}(\tilde{t}_w) = \det(\tilde{t}_w)^k = \text{sgn}(w)^k.$$

Hence the character  $e_{\tilde{\mu}}$  is  $\text{sgn}^{\otimes k}$  (see [2] Lemma 1.2).

## 7. The Case of $SL(n, \mathbb{C})$

In this section, we consider a special class of simple Lie groups,  $SL(n, \mathbb{C})$ . We regard  $G = SL(n, \mathbb{C})$  as a subgroup of  $\tilde{G} = GL(n, \mathbb{C})$ .

Let  $\tilde{T}$  and  $T$  be the groups of all diagonal matrices in  $\tilde{G}$  and  $G$  respectively. The Weyl groups  $W$  of  $\tilde{G}$  and  $G$  are both isomorphic to the symmetric group  $\mathfrak{S}_n$  of degree  $n$ . The normalizer  $\tilde{N}$  (resp.  $N$ ) of  $\tilde{T}$  (resp.  $T$ ) is the group of matrices of  $\tilde{G}$  (resp.  $G$ ) that have precisely one non-zero entry in each row and column. Then  $\tilde{N}$  is the semidirect product of  $\tilde{T}$  by  $W$ .

Take the subgroup of all upper triangular matrices as a Borel subgroup of  $\tilde{G}$  and  $G$ . The equivalence classes of irreducible representations of  $\tilde{G}$  are parametrized by Young diagrams with at most  $n$  rows in such a way that every Young diagram  $(m_1, \dots, m_n)$  corresponds to the representation with highest weight  $\tilde{\rho}$  given by  $\tilde{\rho}(\tilde{t}) = \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$  for  $\tilde{t} = \text{diag}(\tilde{t}_1, \dots, \tilde{t}_n)$ .

Every irreducible representation of  $G$  is obtained by restricting an irreducible representation of  $\tilde{G}$ . Let  $\det$  be the linear representation of  $\tilde{G}$  defined by taking the determinant of matrices. For an irreducible representation  $\tilde{\rho}$  of  $\tilde{G}$  corresponding to a Young diagram  $\tilde{\rho} = (\tilde{\rho}_1, \dots, \tilde{\rho}_n)$  (we use a same symbol for a representation and the corresponding Young diagram) and an integer  $k$ , the irreducible representation  $(\det)^{\otimes k} \otimes \tilde{\rho}$  corresponds to the Young diagram  $\tilde{\rho}[k] = (\tilde{\rho}_1 + k, \dots, \tilde{\rho}_n + k)$ . The restrictions of the representations  $\tilde{\rho}[k]$  to  $G$  define the same irreducible representation of  $G$  for any  $k \in \mathbb{Z}$ . Thus the irreducible representations of  $G$  are parametrized by Young diagram with at most  $n - 1$  rows.

**CONVENTION 7.1.** In the sequel, unless otherwise stated, we use the notation  $\tilde{\rho}$ , for a given irreducible representation  $\rho$  of  $G$ , to denote the uniquely determined Young diagram with at most  $n - 1$  rows such that  $\tilde{\rho}|_G = \rho$ .

Let  $\tilde{\rho}$  be an irreducible representation of  $\tilde{G}$  corresponding to a Young diagram  $\tilde{\rho}$  with at most  $n - 1$  rows. Let

$$\tilde{\mu} = a_1 \omega_{i_1} + \cdots + a_{r-1} \omega_{i_{r-1}} + a_r \omega_n$$

$$(a_1, \dots, a_{r-1} > 0, a_r \geq 0, 1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n - 1),$$



be a dominant weight of  $\tilde{\rho}$  such that  $|\tilde{\mu}| = |\tilde{\rho}|$ , where  $\omega_i$  is a weight corresponding to Young diagram  $(1^i)$  and  $|\lambda|$  denotes the sum of the parts  $\lambda_i$  of a Young diagram  $\lambda = (\lambda_1, \dots, \lambda_n)$ . The stabilizer  $W_{\tilde{\mu}}$  of  $\tilde{\mu}$  is a direct product of symmetric groups:

$$W_{\tilde{\mu}} = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}, \quad (7.1)$$

where  $n_1 = i_1, n_2 = i_2 - i_1, \dots, n_r = n - i_{r-1}$ . Let  $\tilde{P}$  be the corresponding parabolic subgroup of  $\tilde{G}$  containing the Borel subgroup. The Levi subgroup  $\tilde{G}_{\tilde{P}}$  of  $\tilde{P}$  is a product of  $GL(n_i, \mathbf{C})$ :

$$\tilde{G}_{\tilde{P}} = GL(n_1, \mathbf{C}) \times \cdots \times GL(n_r, \mathbf{C}). \quad (7.2)$$

Let

$$\tilde{\rho}|_{\tilde{G}_{\tilde{P}}} = \sum_{\mathbf{v}} c_{\mathbf{v}}^{\tilde{\rho}} \mathbf{v} \quad (7.3)$$

be the decomposition of the restriction of  $\tilde{\rho}$  to  $\tilde{G}_{\tilde{P}}$  into irreducible representations  $\mathbf{v}$  of  $\tilde{G}_{\tilde{P}}$ . Every irreducible representation  $\mathbf{v}$  of  $\tilde{G}_{\tilde{P}}$  is given by a set of Young diagrams  $v^i$ ,  $1 \leq i \leq r$ , with at most  $n_i$  rows, which correspond to irreducible representations of  $GL(n_i, \mathbf{C})$ . The summation is taken over  $\mathbf{v}$ 's with  $\sum_{i=1}^r |v^i| = |\tilde{\rho}|$ . The coefficient  $c_{\mathbf{v}}^{\tilde{\rho}}$  is given by the Littlewood-Richardson rule:

$$c_{\mathbf{v}}^{\tilde{\rho}} = \sum_{(\kappa^1, \dots, \kappa^{r-2})} c_{v^1 \kappa^1}^{\tilde{\rho}} c_{v^2 \kappa^2}^{\kappa^1} \cdots c_{v^{r-1} v^r}^{\kappa^{r-2}}, \quad (7.4)$$

where  $c_{\mu, \nu}^{\rho}$  is the Littlewood-Richardson coefficient. We call this a multiple Littlewood-Richardson coefficient.

By (7.1),  $\tilde{T}$  acts on the weight space of  $\tilde{\mu} = a_1 \omega_{i_1} + \cdots + a_{r-1} \omega_{i_{r-1}} + a_r \omega_n$  as the multiplication by

$$(\tilde{t}_1 \cdots \tilde{t}_{n_1})^{f_1} \cdots (\tilde{t}_{n-n_r+1} \cdots \tilde{t}_n)^{f_r}, \quad \text{diag}(\tilde{t}_1, \dots, \tilde{t}_n) \in \tilde{T}, \quad (7.5)$$

where  $f_i = a_i + \cdots + a_r$ ,  $1 \leq i \leq r$ . That is, the multiplicities of  $f_i$  is equal to  $n_i$  in the Young diagram  $\tilde{\mu}$ .

Thus, by the definition of  $\tilde{\rho}_{\tilde{\mu}}$  (see (6.1)), we have

$$\tilde{\rho}_{\tilde{\mu}} = \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} \mathbf{v}.$$

Let  $\rho = \tilde{\rho}|_G$  and  $\mu$  be the dominant weight of  $\rho$  with  $\tilde{\mu}|_T = \mu$ . Note that the correspondence of the weights  $\tilde{\mu}$  of  $\tilde{\rho}$  to the weights  $\mu$  of  $\rho$  is one-to-one, since the

difference between the highest weight of  $\tilde{\rho}$  (resp.  $\rho$ ) and a weight  $\tilde{\mu}$  of  $\tilde{\rho}$  (resp. a weight  $\mu$  of  $\rho$ ) is a linear combination of roots and  $|\tilde{\mu}| = |\tilde{\rho}|$ . Then  $G_P = \tilde{G}_{\tilde{P}} \cap G$  and

$$G'_P = SL(n_1, \mathbf{C}) \times \cdots \times SL(n_r, \mathbf{C}).$$

Now we have the main result in this section.

**THEOREM 7.2.** *Let  $\rho$  be an irreducible representation of  $SL(n, \mathbf{C})$  and  $\tilde{\rho}$  as in Convention 7.1. For a weight  $\mu$  of  $\rho$ , let  $\tilde{\mu}$  be the weight of  $\tilde{\rho}$  such that  $|\tilde{\mu}| = |\tilde{\rho}|$  and  $\tilde{\mu}|_T = \mu$ . Then the restriction of  $\rho$  to  $N$  decomposes as*

$$\rho|_N = \sum_{\mu \in \Lambda_p^+} \sum_{\tau \in \text{Irr}(N_{\mu})} d_{\mu, \tau} \theta_{\mu, \tau},$$

where

$$d_{\mu, \tilde{\mu} \rtimes \sigma|_{N_{\tilde{\mu}}}} = d_{\mu, \tau(\mu, \sigma')} = \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} d_{\mathbf{v}, \sigma'}^0,$$

$$\sigma' = \sigma \otimes (\text{sgn}^{\otimes f_1} \times \cdots \times \text{sgn}^{\otimes f_r})$$

with  $(f_1, \dots, f_r)$  defined in (7.5) for  $\tilde{\mu}$ .

**PROOF.** By (7.5) and Example 6.6, we have

$$e_{\tilde{\mu}}(w) = \text{sgn}(w_1)^{f_1} \cdots \text{sgn}(w_r)^{f_r},$$

for an element  $w = w_1 \cdots w_r \in W_{\tilde{\mu}}$  ( $w_i \in \mathfrak{S}_{n_i}$ ). Thus we have

$$\sigma \otimes (e_{\tilde{\mu}}|_{W_{\tilde{\mu}}}) = \sigma \otimes (\text{sgn}^{\otimes f_1} \times \cdots \times \text{sgn}^{\otimes f_r}).$$

The result follows from Theorem 6.5. □

## 8. Explicit Formula for the Branching from $SL(n, \mathbf{C})$ to $N$ (I)

We give a multiplicity formula of an irreducible representation of  $N$  in a given irreducible representation of  $G$  for the case  $G = SL(n, \mathbf{C})$ . The notation is as in section 7.

Let  $\mathbf{v} = (v^1, \dots, v^r)$ ,  $|v^i| = f_i n_i$ , be an irreducible representation of  $\tilde{G}_{\tilde{P}}$ . Let  $\zeta_{(v^i)_0}$  be the character of the representation of  $\mathfrak{S}_{n_i}$  induced on the zero weight space of  $v^i|_{SL(n_i, \mathbf{C})}$ . The character value of  $\zeta_{(v^i)_0}$  is calculated as follows. Let  $w_i$  be

an element of  $\mathfrak{S}_{n_i}$  with cyclic factorization  $w_i = w_{i1} \cdots w_{ik_i}$ . Let  $l_{ij}$  be the order of  $w_{ij}$ , then for every  $i$ , the set  $\{l_{ij}\}$  form a uniquely determined partition  $\pi_i$  of  $n_i$ .

Let

$$H_i = GL(l_{i1}, \mathbf{C}) \times \cdots \times GL(l_{ik_i}, \mathbf{C})$$

and

$$v^i|_{H_i} = \sum_{\lambda=(\lambda^1, \dots, \lambda^{k_i})} c_{\lambda}^{v^i} \lambda, \quad (8.1)$$

where  $\lambda = (\lambda^1, \dots, \lambda^{k_i})$  is a sequence of Young diagrams  $\lambda^j$  with at most  $l_{ij}$  rows, which corresponds to an irreducible representation of  $GL(l_{ij}, \mathbf{C})$ . Then we have ([2] §4)

$$\zeta_{(v^i)_0}(w_i) = \sum_{\substack{\lambda=(\lambda^1, \dots, \lambda^{k_i}) \\ |\lambda^j|=l_{ij}f_i}} c_{\lambda}^{v^i} \text{sgn}(w_i)^{f_i k_i} \eta_{(\lambda^1)_0}(w_{i1}) \cdots \eta_{(\lambda^{k_i})_0}(w_{ik_i}), \quad (8.2)$$

where  $\eta_{(\lambda^j)_0}$  is the character of the representation of  $\mathfrak{S}_{l_{ij}}$  induced on the zero weight space of representation  $\lambda^j|_{SL(l_{ij}, \mathbf{C})}$ .

Let  $x_{ij}$  be the permutation matrix of  $GL(l_{ij}, \mathbf{C})$  corresponding to the cycle  $w_{ij}$ . Since  $w_{ij}$  is an  $l_{ij}$ -cycle,  $w_{ij}$  is a Coxeter element of the Weyl group  $\mathfrak{S}_{l_{ij}}$  of  $SL(l_{ij}, \mathbf{C})$ . Hence the trace of  $\lambda^j(x_{ij})$  is equal to  $\text{sgn}(w_{ij})^{f_i} \eta_{(\lambda^j)_0}(w_{ij})$ . In [9], it is shown that the trace of  $\lambda^j(x_{ij}) = 0$  or  $\pm 1$ .

The trace of  $\lambda^j(x_{ij})$  is also calculated by the generalized  $q$ -binomial coefficients as follows. For an integer  $m$  and a Young diagram  $\gamma = (\gamma_1, \dots, \gamma_s)$ , the generalized  $q$ -binomial coefficients in the indeterminate  $q$  is defined to be

$$\begin{bmatrix} m \\ \gamma' \end{bmatrix} = \prod_{x \in \gamma} \frac{1 - q^{m-c(x)}}{1 - q^{h(x)}},$$

where  $\gamma' = (\gamma'_1, \dots, \gamma'_t)$  is the transpose of  $\gamma$ ,  $c(x) = j - i$  is the content and  $h(x) = \gamma_i + \gamma'_j - i - j + 1$  is the hook length for each  $x = (i, j) \in \gamma$  (here we regard  $\gamma$  as a matrix. See e.g. [10] Chap. I §1, Ex. 3, §3, Ex. 1). It is shown that the generalized  $q$ -binomial coefficient is a polynomial in  $q$  and

$$s_{\gamma}(1, q, q^2, \dots, q^{m-1}) = q^{n(\gamma)} \begin{bmatrix} m \\ \gamma' \end{bmatrix},$$

where  $s_{\gamma}$  is the Schur function of  $m$  variables corresponding to  $\gamma$  and  $n(\gamma) = \sum_{i=1}^s (i-1)\gamma_i$ .

On the other hand,  $s_\gamma$  is the restriction to  $T$  of the irreducible character of  $GL(m, \mathbb{C})$  corresponding to  $\gamma$  and the permutation matrix corresponding to a cycle of order  $m$  is conjugate to the diagonal matrix  $\text{diag}(1, \omega, \omega^2, \dots, \omega^{m-1})$ ,  $\omega = e^{2\pi\sqrt{-1}/m}$ . Thus we have

$$\text{sgn}(w_{ij})^{f_i} \eta_{(\lambda^j)_0}(w_{ij}) = q^{n(\lambda^j)} \left[ \begin{matrix} l_{ij} \\ \lambda^{j'} \end{matrix} \right] \Big|_{q=\exp(2\pi\sqrt{-1}/l_{ij})}. \quad (8.3)$$

**THEOREM 8.1.** *For an irreducible representation  $\rho$  of  $G$  and an irreducible representation  $\theta_{\mu, \tau}$  of  $N$ , the multiplicity  $[\rho|_N : \theta_{\mu, \tau}]$  is*

$$\frac{1}{|W_\mu|} \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} \sum_{\substack{w=w_1 \cdots w_r \in W_\mu \\ w_i \in \mathfrak{S}_{n_i}}} \phi_{\sigma'}(w) \prod_{i=1}^r \left( \sum_{\substack{\lambda=(\lambda^1, \dots, \lambda^{k_i}) \\ |\lambda^j|=l_{ij} f_i}} c_{\lambda}^{v^i} \prod_{j=1}^{k_i} b(\lambda^j, l_{ij}) \right),$$

where  $\tilde{\rho}$  is determined from  $\rho$  as in Convention 7.1,  $W_\mu = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$  is the stabilizer of  $\mu$  in  $W$ ,  $c_{\mathbf{v}}^{\tilde{\rho}}$  and  $c_{\lambda}^{v^i}$  are multiple Littlewood-Richardson coefficients ((7.4), (8.1)),  $\phi_{\sigma'}$  is the character of irreducible representation

$$\sigma' = (\text{sgn}^{\otimes f_1} \times \cdots \times \text{sgn}^{\otimes f_r}) \otimes \sigma$$

of  $W_\mu$ , the integer  $f_i$  is given in (7.5),  $\sigma$  is defined from  $\tau$  as in Theorem 5.1,  $l_{ij}$  is the order of cycle of  $w_{ij}$ , and  $b(\lambda^j, l_{ij})$  is the integer given by (8.3).

**PROOF.** Let  $\zeta_{(\mathbf{v})_0}$  be the character of  $W_\mu$ -module  $\mathbf{v}_0$ . By Theorem 7.2 we have

$$\begin{aligned} [\rho|_N : \theta_{\mu, \tau}] &= \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} [(\mathbf{v})_0 : \sigma'] \\ &= \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} (\zeta_{(\mathbf{v})_0}, \phi_{\sigma'}) \\ &= \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} \frac{1}{|W_\mu|} \sum_{w \in W_\mu} \zeta_{(\mathbf{v})_0}(w) \phi_{\sigma'}(w) \\ &= \sum_{\substack{\mathbf{v}=(v^1, \dots, v^r) \\ |v^i|=f_i n_i}} c_{\mathbf{v}}^{\tilde{\rho}} \frac{1}{|W_\mu|} \sum_{\substack{w=w_1 \cdots w_r \in W_\mu \\ w_i \in \mathfrak{S}_{n_i}}} \phi_{\sigma'}(w) \prod_{i=1}^r \zeta_{(\mathbf{v}^i)_0}(w_i). \end{aligned}$$

By (8.2) and (8.3), we have the result.  $\square$

### 9. Explicit Formula for the Branching from $SL(n, \mathbb{C})$ to $N$ (II)

Let  $G_c$  be a compact real form of a complex semisimple Lie group  $G$ ,  $T_c$  a maximal torus of  $G_c$  and  $N_c$  its normalizer in  $G_c$ . Let  $T$  be the maximal torus of  $G$  containing  $T_c$  and  $N$  its normalizer in  $G$ . Then  $T_c = T \cap G_c$  and the Weyl group  $W = N/T$  is isomorphic to  $N_c/T_c$ . Every finite dimensional irreducible continuous representation of  $G_c$  (resp.  $T_c$ ) is obtained from a finite dimensional irreducible holomorphic representation of  $G$  (resp.  $T$ ) by restriction. This gives a one-to-one correspondence between these representations of  $G_c$  and  $G$  (resp.  $T_c$  and  $T$ ). Note that our representatives  $n_w$  of the Weyl group in  $N$  actually lie in  $G_c$  (see Remark 5.4 (i)). By the construction of irreducible representations of  $N$  and  $N_c$ , we have a one-to-one correspondence between the equivalence classes of irreducible representations of  $N$  and those of  $N_c$  by restriction.

In this section we give an explicit formula for the branching from  $SL(n, \mathbb{C})$  to  $N$  in a way different from that of the previous section.

Let  $\tilde{G}$ ,  $\tilde{N}$ ,  $\tilde{T}$  be as in §7. Let  $\tilde{N}_c = \tilde{N} \cap U(n)$  and  $\tilde{T}_c = \tilde{T} \cap U(n)$ .  $\tilde{N}_c$  is a semidirect product of  $\tilde{T}_c$  by  $\mathfrak{S}_n$ . Then we have

$$\text{Irr}(\tilde{N}_c) = \{\theta_{\tilde{\mu}, \tilde{\mu} \times \sigma} \mid \tilde{\mu} \in \tilde{\Lambda}^+, \sigma \in \text{Irr}(W_{\tilde{\mu}})\},$$

where  $\tilde{\Lambda}^+$  is the set of all characters of  $\tilde{T}_c$  which are dominant with respect to  $\Pi$ . Every irreducible representation of  $U(n)$  is afforded by an irreducible module of  $GL(n, \mathbb{C})$  and every irreducible representation of  $\tilde{N}_c$  is also afforded by an irreducible module of  $\tilde{N}$  by taking the restriction. Hence the multiplicities  $[\tilde{\rho}|_{\tilde{N}} : \theta_{\tilde{\mu}, \tilde{\tau}}]$  are equal to those for  $U(n)$  and  $\tilde{N}_c$ .

Since we work entirely within  $U(n)$  in this section, we replace  $\tilde{T}_c$ ,  $\tilde{N}_c$ ,  $\tilde{\mu}$ ,  $\tilde{\tau}$  by  $T$ ,  $N$ ,  $\mu$ ,  $\tau$  to avoid complicated notation.

Let  $dn$  be the Haar measure on  $N$  normalized by  $\int_N dn = 1$ . Let  $\psi_{\mu, \tau}$  be the character of  $\theta_{\mu, \tau}$  and  $S_\rho$  the character of irreducible representation  $\rho$  of  $U(n)$ , whose restriction to  $T$  is Schur function  $s_\rho$ . Then we have

$$[\rho|_N : \theta_{\mu, \tau}] = \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} \int_T \psi_{\mu, \tau}(n_w t) \overline{S_\rho(n_w t)} dt, \quad (9.1)$$

where  $n_w t \in N$  and  $dt$  is the normalized Haar measure on  $T$ .

By Proposition 4.1, we have

$$\psi_{\mu, \tau}(n_w t) = \frac{1}{|W_\mu|} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0) \left( \sum_{x \in \Phi_w^{-1}(w_0)} \mu(t^x) \right), \quad (9.2)$$

where  $\tau = \mu \rtimes \sigma$ . Since  $S_\rho(n_w t)$  is the trace of  $\rho(n_w t)$ , there appear only the terms corresponding to the weights fixed by  $w$ :

$$S_\rho(n_w t) = \sum_{\substack{v=(i_1, \dots, i_n) \\ w(v)=v}} c_v t_1^{i_1} \cdots t_n^{i_n}, \quad (9.3)$$

where  $c_v$  is the trace of  $\rho(n_w)$  on the weight space  $V_v$  of weight  $v$ .

Since

$$\int_T (t_1^{a_1} \cdots t_n^{a_n}) \overline{(t_1^{b_1} \cdots t_n^{b_n})} dt = \begin{cases} 1 & a_i = b_i \ (1 \leq i \leq n) \\ 0 & \text{otherwise,} \end{cases} \quad (9.4)$$

we have

$$\int_T \mu(t^x) \overline{S_\rho(n_w t)} dt = c_{x \cdot \mu}. \quad (9.5)$$

For  $x \in \Phi_w^{-1}(w_0)$ ,  $w$  stabilizes the weight  $x \cdot \mu$ , since  $x^{-1}wx = w_0 \in W_\mu$  and hence  $(wx) \cdot \mu = (xw_0) \cdot \mu = x \cdot \mu$ . If  $x, x' \in \Phi_w^{-1}(w_0)$ , then  $x' = zx$  for some element  $z$  of the centralizer of  $w$ . Since the action of  $z$  transforms  $V_{x \cdot \mu}$  to  $V_{x' \cdot \mu}$  and this transformation commutes with the action of  $w$ , the trace of  $\rho(w)$  on  $V_{x \cdot \mu}$  is equal to that on  $V_{x' \cdot \mu}$ . Hence we have

$$c_{x \cdot \mu} = c_{x' \cdot \mu}.$$

We denote this by  $m(\rho, \mu; w, w_0)$ . Then by (9.1), (9.2), (9.5) and (4.3) we have

$$\begin{aligned} [\rho|_N : \theta_{\mu, \tau}] &= \frac{1}{n!} \frac{1}{|W_\mu|} \sum_{w \in \Xi_n} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0) |Z_{\Xi_n}(w)| m(\rho, \mu; w, w_0) \\ &= \sum_{w \in \Xi_n} \frac{|Z_{\Xi_n}(w)|}{n! |W_\mu|} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0) m(\rho, \mu; w, w_0) \end{aligned} \quad (9.6)$$

The problem is now reduced to determination of  $m(\rho, \mu; w, w_0)$ . Since an element  $n_w t \in N$  is diagonalizable,  $m(\rho, \mu; w, w_0)$  is equal to the Schur function evaluated at the eigenvalues  $\varepsilon_{w, t} = (\varepsilon_1, \dots, \varepsilon_n)$  of  $n_w t$ . To calculate this we use the Jacobi-Trudy identity:

$$\begin{aligned} s_\rho &= \det(h_{\rho_i - i + j})_{1 \leq i, j \leq n} \\ &= \begin{vmatrix} h_{\rho_1} & h_{\rho_1+1} & \cdots & h_{\rho_1-1+n} \\ h_{\rho_2-1} & h_{\rho_2} & \cdots & h_{\rho_2-2+n} \\ \vdots & & \ddots & \vdots \\ h_{\rho_n-n+1} & h_{\rho_n-n+2} & \cdots & h_{\rho_n} \end{vmatrix}, \end{aligned} \quad (9.7)$$

where  $\rho = (\rho_1, \dots, \rho_n)$  and  $h_i$  is the  $i$ th complete symmetric function: the sum of all distinct monomials of degree  $i$ .

We have

$$\begin{aligned}
 m(\rho, \mu; w, w_0) &= \int_T \mu(t^x) \overline{S_\rho(n_w t)} dt \\
 &= \int_T \mu(t^x) \left( \overline{\sum_{\gamma \in \mathfrak{S}_n} \text{sgn}(\gamma) h_{\rho_1-1+\gamma(1)}(\varepsilon_{w,t}) \cdots h_{\rho_n-n+\gamma(n)}(\varepsilon_{w,t})} \right) dt \\
 &= \sum_{\gamma \in \mathfrak{S}_n} \text{sgn}(\gamma) \int_T \mu(t^x) \overline{h_{\rho_1-1+\gamma(1)}(\varepsilon_{w,t}) \cdots h_{\rho_n-n+\gamma(n)}(\varepsilon_{w,t})} dt \quad (9.8)
 \end{aligned}$$

Since  $h_i$  is the restriction to  $T$  of the character of representation of  $GL(n, \mathbf{C})$  on the  $i$ th symmetric tensor of  $\mathbf{C}^n$ , we next study the action of  $n_w t$  on the weight space of  $x \cdot \mu$  in the tensor product  $S^{\rho_1-1+\gamma(1)} \otimes \cdots \otimes S^{\rho_n-n+\gamma(n)}$ , where  $S^k$  is  $k$ th symmetric tensor of  $\mathbf{C}^n$ . Let

$$\{e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

be a basis of  $S^k$ , where  $e_1, \dots, e_n$  are the standard basis of  $\mathbf{C}^n$  and  $e_{i_1} \cdots e_{i_k}$  is the symmetric tensor product of  $e_{i_1}, \dots, e_{i_k}$ .

LEMMA 9.1. *The trace of  $\rho(n_w t)$  on the space of weight  $x \cdot \mu$  is equal to that of  $\rho(n_{w_0} t)$  on the space of weight  $\mu$ .*

PROOF. This follow from  $x^{-1} w x = w_0$ . □

By this lemma, we may consider the action of  $w_0$  on the space of weight  $\mu$  in the tensor product  $S^{q_1} \otimes \cdots \otimes S^{q_n}$ .

Recall  $\mu(t) = (t_1 \cdots t_{n_1})^{f_1} \cdots (t_{n-n_r+1} \cdots t_n)^{f_r}$  (see (7.5)) and  $W_\mu = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$ . Write  $w_0 = w_1 \cdots w_r$  with  $w_i \in \mathfrak{S}_{n_i}$ . Let  $w_i = w_{i1} \cdots w_{ik_i}$  be a cyclic factorization and  $l_{ij}$  the order of  $w_{ij}$ .

Let  $(1^{p_{i1}}, \dots, n_i^{p_{in_i}})$  be the cycle type of  $w_i$  ( $1 \leq i \leq r$ ). Then  $n_i = \sum_{j=1}^{n_i} j p_{ij}$ . The set

$$\left\{ (e_1^{\alpha_{11}} \cdot e_2^{\alpha_{12}} \cdots e_n^{\alpha_{1n}}) \otimes \cdots \otimes (e_1^{\alpha_{n1}} \cdot e_2^{\alpha_{n2}} \cdots e_n^{\alpha_{nn}}) \left| \begin{array}{l} q_i = \sum_{j=1}^n \alpha_{ij} \\ \mu_j = \sum_{i=1}^n \alpha_{ij} \end{array} \right. \right\} \quad (9.9)$$

gives a basis of the space of weight  $\mu$  in  $S^{q_1} \otimes \cdots \otimes S^{q_n}$ , where  $\mu(t) = t_1^{\mu_1} \cdots t_n^{\mu_n}$ . The action of  $w_0$  on the weight space induces a permutation of the columns of the  $n \times n$  matrix  $(\alpha_{ij})$ . Hence the condition that  $w_0$  fixes an element of the basis is equivalent to the condition

$$\alpha_{sj_1} = \alpha_{sj_2} = \cdots, \quad 1 \leq s \leq n \quad (9.10)$$

for any cycle  $(j_1, j_2, \dots)$  of  $w_i$ ,  $1 \leq i \leq r$ . Thus we put these integers determined for cycles of  $w_i$  in the form of  $n \times p_i$  matrix

$$A_i = (a_{st}^i)_{\substack{1 \leq s \leq n, \\ 1 \leq t \leq p_i}}, \quad 1 \leq i \leq r,$$

where  $p_i = \sum_{j=1}^{n_i} p_{ij}$  is the number of cycles of  $w_i$ .

Let

$$b_s^i = a_{s1}^i l_{i1} + \cdots + a_{sp_i}^i l_{ik_i}, \quad 1 \leq i \leq r, 1 \leq s \leq n.$$

Then the integers  $a_{st}^i$  and  $b_s^i$  satisfy the conditions

$$f_i = \sum_{k=1}^n a_{kt}^i, \quad 1 \leq i \leq r, 1 \leq t \leq p_i \quad (9.11)$$

$$q_s = \sum_{k=1}^r b_s^k, \quad 1 \leq s \leq n. \quad (9.12)$$

EXAMPLE 9.2. Let  $n = 5$ ,  $(q_1, q_2) = (5, 3)$  and  $\mu(t) = (t_1 t_2 t_3)^2 (t_4 t_5) : r = 2$ ,  $(f_1, f_2) = (2, 1)$ . Put

$$w_0 = (1)(23)(45), \quad w_1 = (1)(23), \quad w_2 = (45).$$

The matrices  $(\alpha_{ij})$  corresponding to the vector of weight  $\mu$  in  $S^5 \otimes S^3$  are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \dots,$$

where we omit the last 3 rows. Among them,  $w_0$  fixes only the above two matrices. Hence for the former we have

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (b_1^1, b_2^1) = (3, 3), \quad (b_1^2, b_2^2) = (2, 0),$$

and for the latter

$$A_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (b_1^1, b_2^1) = (5, 1), \quad (b_1^2, b_2^2) = (0, 2).$$



LEMMA 9.3.

$$m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0) = \# \left\{ (A_1, \dots, A_r) \left| \begin{array}{l} A_i = (a_{st}^i)_{\substack{1 \leq s \leq n \\ 1 \leq t \leq p_i}}, \quad 1 \leq i \leq r \\ f_i = \sum_{k=1}^n a_{kt}^i, \quad 1 \leq i \leq r, 1 \leq t \leq p_i \\ q_s = \sum_{k=1}^r b_s^k, \quad 1 \leq s \leq n \end{array} \right. \right\} \quad (9.13)$$

PROOF. The action of  $W_\mu$ , the stabilizer of the weight  $\mu$ , on  $S^{q_1} \otimes \cdots \otimes S^{q_n}$  induces permutation of the elements  $u$  of the set given by (9.9), which also induces a permutation of columns of the matrix  $(\alpha_{ij})$ . Hence the number of elements  $u$  of weight  $\mu$  that are fixed by  $w_0$  is equal to the number of the set of the matrices  $(A_1, \dots, A_r)$  satisfying (9.11) and (9.12), which is also equal to  $m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0)$  by Lemma 9.1.  $\square$

REMARK 9.4. If  $w_0, w'_0 \in W_\mu \cap K_w$  are conjugate in  $W_\mu$ , then

$$m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0) = m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w'_0).$$

We come to give a multiplicity formula.

THEOREM 9.5. Let  $\rho$  be an irreducible representation of  $U(n)$  and  $\theta_{\mu, \tau}$  an irreducible representation of  $N$ , then

$$\begin{aligned} [\rho|_N : \theta_{\mu, \tau}] &= \sum_{w \in \mathfrak{S}_n} \frac{|Z_{\mathfrak{S}_n}(w)|}{n! |W_\mu|} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0) \\ &\quad \times \sum_{\gamma \in \mathfrak{S}_n} \text{sgn}(\gamma) m(S^{\rho_1 - 1 + \gamma(1)} \otimes \cdots \otimes S^{\rho_n - n + \gamma(n)}, \mu; w, w_0), \end{aligned}$$

where  $\tau = \mu \rtimes \sigma \in \text{Irr}(N_\mu)$  (see Remark 2.4). The number  $m(S^{\rho_1 - 1 + \gamma(1)} \otimes \cdots \otimes S^{\rho_n - n + \gamma(n)}, \mu; w, w_0)$  is given by (9.13).

PROOF. The result follows from (9.6), (9.8), Lemma 9.1 and 9.3.  $\square$

## 10. On the Irreducibility of the $N$ -Span of a Weight Space for $GL(n, \mathbf{C})$

Every irreducible module of  $\mathfrak{S}_n$  is afforded by the zero weight space of certain irreducible module of  $SL(n, \mathbf{C})$  ([8], [9], [2]). We consider the similar

problem for  $N$  and give a series of examples of irreducible modules  $V$  of  $GL(n, \mathbf{C})$  and their weights  $\mu$  such that the  $N$ -modules

$$\bigoplus_{w \in W/W_\mu} V_{w(\mu)}$$

are irreducible, where the summation is taken over a complete set of coset representatives of  $W/W_\mu$ . We denote this  $N$ -module by  $V_{W_\mu}$ .

Let  $G = GL(n, \mathbf{C})$ ,  $T$  the group of all diagonal matrices in  $G$ , and  $N$  the normalizer of  $T$  in  $G$ , which is isomorphic to the semidirect product of  $T$  by the Weyl group  $W \simeq \mathfrak{S}_n$ . Let  $\rho$  be an irreducible rational representation of  $G$  afforded by  $V$  and  $V_\mu$  the weight space with dominant weight  $\mu$  of  $T$  given by

$$\mu(t) = (t_1 \cdots t_{n_1})^{f_1} (t_{n_1+1} \cdots t_{n_1+n_2})^{f_2} \cdots (t_{n-n_r+1} \cdots t_n)^{f_r},$$

where  $t = (t_1, \dots, t_n) \in T$  and  $f_1 > \cdots > f_r \geq 0$ ,  $n_1 + \cdots + n_r = n$ . We regard  $\mu$  as the Young diagram

$$(\underbrace{f_1, \dots, f_1}_{n_1}, \underbrace{f_2, \dots, f_2}_{n_2}, \dots, \underbrace{f_r, \dots, f_r}_{n_r}).$$

The stabilizer subgroup  $W_\mu$  of  $\mu$  in  $W$  is isomorphic to  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$ . Let  $\sigma^i = (\sigma_1^i, \dots, \sigma_{n_i}^i)$  be a young diagram with  $|\sigma^i| = n_i$ , which corresponds to an irreducible representation of  $\mathfrak{S}_{n_i}$ . Then  $\sigma = (\sigma^1, \dots, \sigma^r)$  corresponds to an irreducible representation of  $W_\mu$ . We will give a Young diagram corresponding to an irreducible representation of  $G$  whose restriction to  $N$  contains the irreducible representation  $\theta_{\mu, \mu \rtimes \sigma}$  of  $N$ .

We add rectangular diagram of length  $n_i$  to  $\sigma^i$  or  $(\sigma^i)'$ :

$$\gamma^i = \begin{cases} \sigma^i + \underbrace{(f_i, \dots, f_i)}_{n_i} & f_i \text{ is odd,} \\ (\sigma^i)' + \underbrace{(f_i, \dots, f_i)}_{n_i} & f_i \text{ is even,} \end{cases} \quad (10.1)$$

where  $(\sigma^i)'$  is the transpose of  $\sigma^i$ . Put its transpose



$$(\gamma^i)' = (\gamma_1^{i'}, \gamma_2^{i'}, \dots)$$

and define a new Young diagram  $\Gamma(\mu, \sigma)'$  by adding up  $j$ th columns of  $\gamma^1, \dots, \gamma^r$  for each  $j$ :

$$\Gamma(\mu, \sigma)' = (\hat{s}_1, \hat{s}_2, \dots), \quad \hat{s}_j = \sum_{i=1}^r \gamma_j^{i'}.$$

Denote its transpose by  $\Gamma(\mu, \sigma)$ .

$$\sigma^1: \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \quad \sigma^2: \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

$\gamma^1$ :   $\gamma^2$ : 

$\Gamma(\mu, \sigma)$

LEMMA 10.3. *Let  $\rho$  be an irreducible representation  $(\det)^{-1} \otimes \Gamma(\mu, \sigma)$  of  $G$  and  $V_\mu$  the weight space with weight  $\mu$ . Then  $\theta_{\mu, \mu \rtimes \sigma}$  is a subrepresentation of  $V_{W_\mu}$ .*

$$t^{(1)} = (t_1, \dots, t_{n_1}), t^{(2)} = (t_{n_1+1}, \dots, t_{n_1+n_2}), \dots, t^{(r)} = (t_{n-n_r+1}, \dots, t_n),$$
$$S_{\Gamma(\mu, \sigma)}(t) = \sum_{\mathbf{v}=(v^{(0)}, \dots, v^{(r)})} S_{v^{(1)}/v^{(0)}}(t^{(1)}) S_{v^{(2)}/v^{(1)}}(t^{(2)}) \cdots S_{v^{(r)}/v^{(r-1)}}(t^{(r)}), \quad (10.2)$$

where the summation is taken over all sequences  $(v^{(0)}, \dots, v^{(r)})$  of Young diagrams such that  $v^{(0)} = \emptyset$ ,  $v^{(r)} = \Gamma(\mu, \sigma)$  and  $v^{(0)} \subset v^{(1)} \subset \dots \subset v^{(r)}$ . The skew Schur function is a sum of the Schur functions:

$$S_{v^{(i)}/v^{(i-1)}} = \sum_{\lambda} c_{v^{(i-1)}\lambda}^{v^{(i)}} S_{\lambda}, \quad (10.3)$$

where  $c_{v^{(i-1)}\lambda}^{v^{(i)}}$  is the Littlewood-Richardson coefficient. Hence we have

$$\begin{aligned} & (t_1 \cdots t_n)^{-1} S_{\Gamma(\mu, \sigma)}(t) \\ &= \sum_{\mathbf{v}=(v^{(0)}, \dots, v^{(r)})} \{(t_1 \cdots t_{n_1})^{-1} S_{v^{(1)}/v^{(0)}}(t^{(1)})\} \cdots \{(t_{n-n_r+1} \cdots t_n)^{-1} S_{v^{(r)}/v^{(r-1)}}(t^{(r)})\}. \end{aligned} \quad (10.4)$$

This gives the decomposition of the restriction  $\det^{-1} \otimes \Gamma(\mu, \sigma)|_{G_{\mu}}$ . Here  $G_{\mu}$  is the Levi subgroup of the standard parabolic subgroup corresponding to  $W_{\mu}$ :

$$G_{\mu} = GL(n_1, \mathbf{C}) \times \cdots \times GL(n_r, \mathbf{C}).$$

As in § 6, let  $(\det^{-1} \otimes \Gamma(\mu, \sigma))_{\mu}$  be the sum of all irreducible subrepresentations of  $\det^{-1} \otimes \Gamma(\mu, \sigma)|_{G_{\mu}}$  having  $\mu$  as its weight. Then the character of  $(\det^{-1} \otimes \Gamma(\mu, \sigma))_{\mu}$  is given by

$$\sum_{\substack{\mathbf{v}=(v^{(0)}, \dots, v^{(r)}) \\ |v^{(i)}/v^{(i-1)}|=(f_i+1)n_i}} \{(t_1 \cdots t_{n_1})^{-1} S_{v^{(1)}/v^{(0)}}(t^{(1)})\} \cdots \{(t_{n-n_r+1} \cdots t_n)^{-1} S_{v^{(r)}/v^{(r-1)}}(t^{(r)})\}. \quad (10.5)$$

Put

$$\hat{s}_j^{(k)} = \sum_{i=1}^k \gamma_j^{i'}$$

and define  $\Gamma^{(k)}(\mu, \sigma)'$  by

$$\Gamma^{(k)}(\mu, \sigma)' = (\hat{s}_1^{(k)}, \hat{s}_2^{(k)}, \dots), \quad \Gamma^{(0)}(\mu, \sigma)' = \emptyset.$$

Denote by  $\Gamma^{(k)}(\mu, \sigma)$  its transpose. Then we have a sequence of Young diagrams:

$$\emptyset = \Gamma^{(0)}(\mu, \sigma) \subset \Gamma^{(1)}(\mu, \sigma) \subset \cdots \subset \Gamma^{(r)}(\mu, \sigma) = \Gamma(\mu, \sigma),$$

which appears in the summation (10.5), since

$$|\Gamma^{(i)}(\mu, \sigma)/\Gamma^{(i-1)}(\mu, \sigma)| = (f_i + 1)n_i.$$

Note that this is equal to  $|\gamma^i|$ . We next show

$$c_{\Gamma^{(i-1)}(\mu, \sigma)}^{\Gamma^{(i)}(\mu, \sigma)} \gamma^i \neq 0. \quad (10.6)$$

The number  $c_{\Gamma^{(i-1)}(\mu, \sigma)}^{\Gamma^{(i)}(\mu, \sigma)} \gamma^i$  is given by the Littlewood-Richardson rule (see [10]): the number of column-strict skew tableaux  $T$  of shape  $\Gamma^{(i)}(\mu, \sigma) - \Gamma^{(i-1)}(\mu, \sigma)$  and of weight  $\gamma^i$  such that the word  $w(T)$  is a lattice permutation. Here the word  $w(T)$  is obtained by reading the entries of  $T$  from right to left in each row, starting with the top row and proceeding downward. The word  $w(T) = (a_1, \dots, a_k)$  is called a lattice permutation if for  $1 \leq j \leq k$ , in the first  $j$  elements of  $w(T)$ , the number of occurrence of  $i$  is not less than the number of occurrence of  $i+1$  for each  $i$ .

By the definition of  $\gamma^i$  and  $\Gamma^{(i)}(\mu, \sigma)$ , if we fill each of the columns of the skew diagram  $\Gamma^{(i)}(\mu, \sigma) - \Gamma^{(i-1)}(\mu, \sigma)$  with the numbers

$$b_1 = n_1 + n_2 + \dots + n_{i-1} + 1, \dots, b_{n_i} = n_1 + \dots + n_i$$

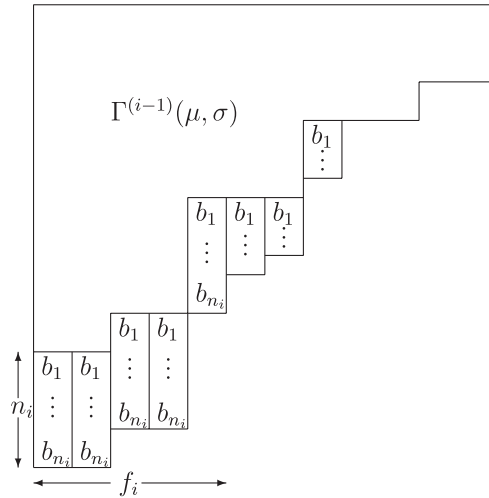


Figure 1: skew tableau  $\Gamma^{(i)}(\mu, \sigma) - \Gamma^{(i-1)}(\mu, \sigma)$

from the top to the bottom in that order, then we have a column-strict skew tableau  $T$  of shape  $\Gamma^{(i)}(\mu, \sigma) - \Gamma^{(i-1)}(\mu, \sigma)$  and of weight  $\gamma^i$ . Since the number  $b_i$  ( $i > 1$ ) lies just below the number  $b_{i-1}$  in  $T$ , the number of the occurrence of  $b_{i-1}$  is not less than that of  $b_i$  in  $w(T)$ , the word  $w(T)$  is a lattice permutation (see Figure 1). Thus we have (10.6).

It follows from (10.3), (10.5) and (10.6) that the irreducible representation  $(\det^{-1} \otimes \gamma^1) \times \dots \times (\det^{-1} \otimes \gamma^r)$  of  $G_\mu$  is a subrepresentation of  $(\det^{-1} \otimes \Gamma(\mu, \sigma))_\mu$ .

By Proposition 10.2 the representation of  $\mathfrak{S}_{n_i}$  induced on the weight space with weight  $(f_i, \dots, f_i)$  ( $n_i$  times) of  $\det^{-1} \otimes \gamma^i$  is equivalent to

$$\begin{cases} \text{sgn}^{\otimes f_i+1} \otimes \sigma^i & f_i \text{ is odd,} \\ \text{sgn}^{\otimes f_i+1} \otimes (\sigma^i)' & f_i \text{ is even} \end{cases}$$

which is equivalent to the irreducible representation of  $\mathfrak{S}_{n_i}$  corresponding to  $\sigma^i$  in both cases. Hence the representation of  $W_\mu$  induced on the weight space with weight  $\mu$  of the representation  $(\det^{-1} \otimes \gamma^1) \times \dots \times (\det^{-1} \otimes \gamma^r)$  of  $G_\mu$  affords the irreducible representation  $\sigma$  as a subrepresentation and the lemma is proved.  $\square$

**PROPOSITION 10.4.** *Let  $V_\mu$  be the weight space with weight  $\mu$  of the representation  $\det^{-1} \otimes \Gamma(\mu, \sigma)$  of  $GL(n, \mathbf{C})$ . If  $\gamma_{n_i}^i \geq \gamma_1^{i+1}$  for  $1 \leq i \leq r-1$ , then  $V_{W_\mu}$  is the irreducible module of  $N$  affording  $\theta_{\mu, \mu \rtimes \sigma}$ .*

**PROOF.** It follows from Lemma 10.3 that  $\theta_{\mu, \mu \rtimes \sigma}$  is a subrepresentation of  $V_{W_\mu}$ . We have only to show  $\dim V_\mu = \deg \sigma$ .

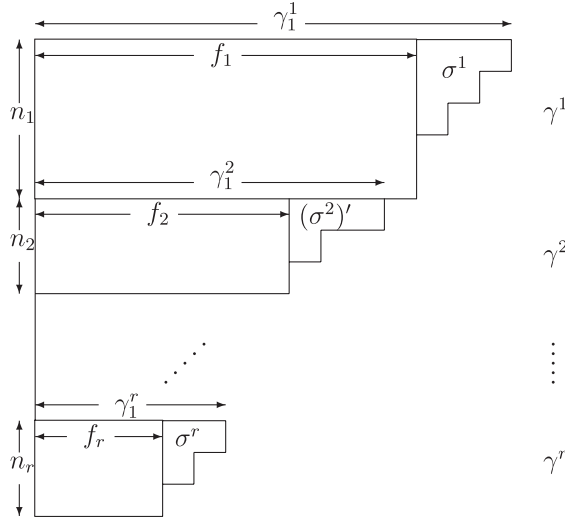
The dimension of  $V_\mu$  is equal to the number of the column-strict tableaux of shape  $\Gamma(\mu, \sigma)$  and of weight  $\mu + (1, \dots, 1)$  ( $n$  times). Put

$$I_1 = \{1, 2, \dots, n_1\}, I_2 = \{n_1 + 1, \dots, n_1 + n_2\}, \dots, I_r = \{n - n_r + 1, \dots, n\}.$$

Since the length  $\gamma_{n_i}^i$  of the last row of  $\gamma^i$  is greater than or equal to the length  $\gamma_1^{i+1}$  of the first row of  $\gamma^{i+1}$ , the diagram  $\Gamma(\mu, \sigma)$  is obtained simply by putting  $\gamma^i$  on  $\gamma^{i+1}$  (Figure 2).

The integer  $i$  in the column-strict tableau should be in the first  $i$  rows of  $\Gamma(\mu, \sigma)$ . Thus the elements of  $I_1$  should be placed in the first  $n_1$  rows, whose shape is just  $\gamma^1$ . Since  $|\gamma^i| = (f_i + 1)n_i$  and every element of  $I_1$  appears exactly  $f_i + 1$  times in  $T$ , the first  $n_1$  rows are filled only with the numbers of  $I_1$  and any element of  $I_1$  doesn't occur in the  $i$ th rows for  $i > n_1$ . The elements of  $I_2$  should be placed in the next  $n_2$  rows, whose shape is  $\gamma^2$ , and so on.

The diagram of  $\gamma^i$  is obtained by adjoining that of  $\sigma^i$  or  $(\sigma^i)'$  to the right of a rectangular diagram  $(f_i, \dots, f_i)$  ( $n_i$  times). The first  $f_i$  columns of  $\gamma^i$  are in the rectangular part and filled with all member of  $I_i$ . Since every element of  $I_i$  appears exactly  $f_i$  times in the rectangular part of  $\gamma^i$ , the number of column-strict tableaux of shape  $\gamma^i$  and of weight  $(f_i + 1, \dots, f_i + 1)$  ( $n_i$  times) is equal to the number of those of shape  $\sigma^i$  (or  $(\sigma^i)'$ ) and of weight  $(1, \dots, 1)$  ( $n_i$  times), which is nothing but the degree of irreducible representation of  $\mathfrak{S}_{n_i}$  corresponding to the Young diagram  $\sigma^i$  (or  $(\sigma^i)'$ ). Hence we have

Figure 2:  $\Gamma(\mu, \sigma)$ 

$$\dim V_\mu = \prod_{i=1}^r \deg \sigma^i = \deg \sigma,$$

which finishes the proof.  $\square$

We show another examples of irreducible representations of  $G$  and their weights  $\mu$  such that  $V_{W_\mu}$  affords  $\theta_{\mu, \mu \rtimes \sigma}$ .

For a Young diagram  $\lambda = (\lambda_1, \dots, \lambda_m)$  with  $|\lambda| = m$  and an integer  $d$  with  $d \geq \lambda_1$ , define a Young diagram  $\lambda^*(d)$  by

$$\lambda^*(d) = (d - \lambda_m, d - \lambda_{m-1}, \dots, d - \lambda_1).$$

**REMARK 10.5.** The contragradient representation of the representation of  $GL(m, \mathbb{C})$  corresponding to  $\lambda$  is  $(\det^{-1})^{\otimes d} \otimes \lambda^*(d)$ .

**PROPOSITION 10.6** ([2], Proposition 5.1). *Let  $\mu = (d - 1, \dots, d - 1)$  ( $m$  times) and  $U_\mu$  be the weight space with weight  $\mu$  of the irreducible representation of  $GL(m, \mathbb{C})$  corresponding to the Young diagram  $\lambda^*(d)$ . The representation of  $\mathfrak{S}_m$  induced on  $U_\mu$  is equivalent to  $(\text{sgn})^{\otimes d} \otimes \lambda$ , where  $\lambda$  represents the irreducible representation of  $\mathfrak{S}_m$  corresponding to the Young diagram  $\lambda$ .*

For a weight  $\mu$  and Young diagrams  $\sigma^i = (\sigma_1^i, \sigma_2^i, \dots)$  ( $1 \leq i \leq r$ ), let  $a$  be the smallest number of the set

$$\{\{f_i + 1 - \sigma_1^i \mid f_i \text{ is odd}, 1 \leq i \leq r\} \cup \{f_i + 1 - (\sigma^i)_1' \mid f_i \text{ is even}, 1 \leq i \leq r\}\},$$

where  $(\sigma^i)'$  is the transpose of  $\sigma^i$ . Put  $e = -a$  (if  $a < 0$ ) and  $e = 0$  (if  $a \geq 0$ ) so that  $f_i + 1 + e - \sigma_1^i \geq 0$  (if  $f_i$  is odd) and  $f_i + 1 + e - (\sigma^i)_1' \geq 0$  (if  $f_i$  is even). Then we can define a Young diagram  $\tilde{\gamma}^i$  by

$$\tilde{\gamma}^i = \begin{cases} (\sigma^i)^*(f_i + 1 + e) & f_i \text{ is odd,} \\ ((\sigma^i)')^*(f_i + 1 + e) & f_i \text{ is even.} \end{cases}$$

For the diagrams  $\tilde{\gamma}^i$  ( $1 \leq i \leq r$ ), we define a Young diagram  $\tilde{\Gamma}(\mu, \sigma)$  as  $\Gamma(\mu, \sigma)$ .

EXAMPLE 10.7. Let  $n = 7$ ,  $(n_1, n_2) = (4, 3)$ ,  $\mu = (2, 2, 2, 2, 1, 1, 1)$ ,  $(f_1, f_2) = (2, 1)$  and  $\sigma^1 = (2, 1, 1)$ ,  $\sigma^2 = (2, 1)$ . Then  $e = 0$ .

$$\sigma^1: \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array} \quad \sigma^2: \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

Then  $\tilde{\gamma}^1 = (3, 3, 2)$ ,  $\tilde{\gamma}^2 = (2, 1)$ .

$$\tilde{\gamma}^1: \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \quad \tilde{\gamma}^2: \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

Thus we have  $\tilde{\Gamma}(\mu, \sigma) = (3, 3, 2, 2, 1)$ :

$$\tilde{\Gamma}(\mu, \sigma): \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

PROPOSITION 10.8. Let  $V_\mu$  be the weight space with weight  $\mu$  of the irreducible representation  $\det^{-e} \otimes \tilde{\Gamma}(\mu, \sigma)$  of  $G$ .

- (i)  $\theta_{\mu, \mu \rtimes \sigma}$  is a subrepresentation of  $V_{W\mu}$ .
- (ii) If  $\tilde{\gamma}_{n_i}^i \geq \tilde{\gamma}_1^{i+1}$  for  $1 \leq i \leq r-1$ , then  $V_{W\mu}$  is the irreducible module of  $N$  affording  $\theta_{\mu, \mu \rtimes \sigma}$ .

PROOF. (i) By the same argument of the proof of Lemma 10.3, the irreducible representation  $(\det^{-e} \otimes \tilde{\gamma}^1) \times \cdots \times (\det^{-e} \otimes \tilde{\gamma}^r)$  of  $G_\mu$  is a subrepresentation of  $((\det^{-e}) \otimes \tilde{\Gamma}(\mu, \sigma))_\mu$ .



It follows from Proposition 10.6 that the representation of  $\mathfrak{S}_{n_i}$  induced on the weight space with weight  $(f_i, \dots, f_i)$  ( $n_i$  times) of  $\tilde{\gamma}^i$  is equivalent to

$$\begin{cases} (\text{sgn})^{\otimes f_i+1} \otimes \sigma^i & f_i \text{ is odd,} \\ (\text{sgn})^{\otimes f_i+1} \otimes (\sigma^i)' & f_i \text{ is even.} \end{cases} \quad (10.7)$$

In both cases the representations are equivalent to the irreducible representation of  $\mathfrak{S}_{n_i}$  corresponding to  $\sigma^i$ . Then the representation of  $W_\mu$  induced on the weight space with weight  $\mu$  of the representation  $(\det^{-e} \otimes \tilde{\gamma}^1) \times \cdots \times (\det^{-e} \otimes \tilde{\gamma}^r)$  of  $G_\mu$  affords the irreducible representation  $\sigma$  as a subrepresentation. Hence  $V_{W_\mu}$  affords  $\theta_{\mu, \mu \rtimes \sigma}$  as its subrepresentation.

(ii) Since  $\theta_{\mu, \mu \rtimes \sigma}$  is a subrepresentation of  $V_{W_\mu}$  by (i), we have only to show  $\dim V_\mu = \deg \sigma$ . By the assumption and the same argument of the proof in Proposition 10.4, we have to show that the number of the column-strict tableau of shape  $\tilde{\gamma}^i$  and of weight  $(f_i + e, \dots, f_i + e)$  ( $n_i$  times) is equal to the degree of the irreducible representation  $\sigma^i$  of  $\mathfrak{S}_{n_i}$ . However this follows from Proposition 10.6.  $\square$

REMARK 10.9. The representation  $\tilde{\Gamma}(\mu, \sigma)$  in the Example 10.7 satisfies the condition of the Proposition 10.8 (ii), but not that of Proposition 10.4.

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